A CLASS OF POLYNOMIALS CONNECTED WITH ORTHOGONAL POLYNOMIALS

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Dedicated to Professor Akitsugu Kawaguchi on his 70th birthday

Recently M. Parodi [**1**][1] has obtained some results concerning the Legendre polynomials. This paper develops an analogous idea in the direction on orthogonal polynomials. Several special cases of classical orthogonal polynominals will be considered too.

Let V be a vector space of polynomials of degree less or equal to n. It is known that the orthogonal polynomials $p_n(x)$ form a base B on V. We shall construct a class of polynomials which belong to V, expressed in B and have a given zero $\lambda \in R$ or C, while the other zeros are real located in the interval $[a, b]$ which are the zeros of the orthogonal polynomials $p_n(x)$.

Every orthogoal set of polynomials $p_n(x)$ possesses the three-term recurrence relation [**2**]

$$p_n(x) = (A_n x + B_n)p_{n-1}(x) - C_n p_{n-2}(x)\,, \qquad n = 2, 3, 4, \ldots .$$

A_n, B_n and C_n are constants, $A_n > 0$ and $C_n > 0$ which is valid also for $n = 1$ if we write $p_{-1}(x) = 0$.

Consider the polynomial of degree $n\,(n > 2)$

$$Q_n(x) = \begin{vmatrix} A_n x + C_n - \lambda A_n & -C_n & 0 & \cdots & 0 \\ \omega_{n-1} - 1 & A_{n-1}x + B_{n-1} & -C_{n-1} & \cdots & 0 \\ \omega_{n-2} & -1 & A_{n-2}x + B_{n-2} & \cdots & 0 \\ \omega_{n-3} & 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \omega_2 & 0 & 0 & \cdots & -C_2 \\ \omega_1 & 0 & 0 & \cdots & A_1 x + B_1 \end{vmatrix},$$

where

$$\omega_k = \omega_k(\lambda) = 1 + C_k - B_k - \lambda A_k\,, \qquad k = 2, 3, \ldots, n\,,$$
$$\omega_1 = \omega_1(\lambda) = 1 - B_1 - \lambda A_1\,.$$

If we add to the first column of the determinant the next to the last column we remark that it has a common factor $x - \lambda$.

A simple developing of the determinant gives

$$(1) \qquad Q_n(x) = p_n(x) + (\omega_n - 1)p_{n-1}(x) + \omega_{n-1}C_n p_{n-2}(x) + \omega_{n-2}C_nC_{n-1}p_{n-3}(x) + \cdots + \omega_1 C_n C_{n-1}\ldots C_2 p_0(x)\,.$$

The determinant can be transformed by adding to the first column each of the other ones. In this way we obtain

$$(2) \qquad Q_n(x) = (x - \lambda)(A_n p_{n-1}(x) + A_{n-1}C_n p_{n-2}(x) + A_{n-2}C_nC_{n-1}p_{n-3}(x) + \cdots + A_1 C_n C_{n-1}\ldots C_2 p_0(x))\,.$$

From (1) and (2) it follows the identity

$$(3) \qquad p_n(x) + (\omega_n - 1)p_{n-1}(x) + \omega_{n-1}C_n p_{n-2}(x) + \cdots + \omega_1 C_n \ldots C_2 p_0(x) = (x - \lambda)(A_n p_{n-1}(x) + A_{n-1}C_n p_{n-2}(x) + \cdots)\,.$$

The zeros of $Q_n(x)$ different from λ are the same of the polynomial

1) Numbers in brackets refer to the references at the end of the paper.

$$\psi_{n-1}(x) = A_n p_{n-1}(x) + A_{n-1}C_n p_{n-2}(x) + \cdots + A_1 C_n \ldots C_2 p_0(x) .$$

If we take

$$\omega_k(\lambda) = 0 , \qquad k = 2, 3, \ldots, n ,$$
$$\omega_1(\bar{\lambda}) = C_1 ,$$

the identity (3) yields

$$(4) \qquad p_n(x) - p_{n-1}(x) = (x - \bar{\lambda})(A_n p_{n-1}(x) + A_{n-1}C_n p_{n-2}(x) + A_{n-2}C_n C_{n-1} p_{n-3}(x) + \cdots) .$$

The zeros of the polynomial $p_n(x) - p_{n-1}(x)$ lie in the interior of $[a, b]$ [**2**]. It follows that the zeros of $\psi_{n-1}(x)$ are in $[a, b]$ too, and consequently the zeros of $Q_n(x)$ which are different from λ.

According to (2) and (4) $Q_n(x)$ may be written into the form

$$(5) \qquad Q_n(x) = \frac{x - \lambda}{x - \bar{\lambda}}(p_n(x) - p_{n-1}(x)) .$$

Special case. Let

$$A_n = \frac{(\beta + 2n)(\beta + 2n - 1)}{2n(\beta + n)} , \qquad B_n = \frac{-\beta^2(\beta + 2n - 1)}{2n(\beta + n)(\beta + 2n - 2)} ,$$
$$C_n = \frac{(n - 1)(\beta + 2n - 1)(\beta + n - 1)}{n(\beta + n)(\beta + 2n - 2)} , \qquad \beta > -1 .$$

Then the orthogonal polynomial $p_n(x)$ is the Jacobi polynomial of the form $P_n^{(0,\beta)}(x)$ and $a = -b = -1$, $\lambda = 1$, $P_0^{(0,\beta)} = 1$.

The relation (4) becomes

$$(6) \qquad P_n^{(0,\beta)}(x) - P_{n-1}^{(0,\beta)}(x) = \frac{(x - 1)(\beta + 2n)}{2n(\beta + n)} \sum_{k=1}^{n} (\beta + 2n - 2k + 1)P_{n-k}^{(0,\beta)}(x) ,$$

and (5)

$$(7) \qquad Q_n^J(x) = \frac{x - \lambda}{x - 1}(P_n^{(0,\beta)}(x) - P_{n-1}^{(0,\beta)}(x)) .$$

As a particular case for $\beta = 0$ we have the Legendre polynomial $P_n(x)$ for which we obtain from (6) and (7) the formulas of M. Parodi.

Application. Some definite integrals involving Jacobi and Legendre polynomials may be of interest.

Using the above relations we obtain

$$1^\circ. \quad \int_{-1}^{1} \frac{x - \lambda}{x - 1}(P_n^{(0,\beta)}(x) - P_{n-1}^{(0,\beta)}(x))P_k(x)dx$$
$$= S_n^k - S_{n-1}^k - \frac{(\beta + 2n)(\lambda - 1)}{2n(\beta + n)} \sum_{r=1}^{n-k} (\beta + 2n - 2r + 1)S_{n-r}^k ,$$

$0 \le k \le n$; for $k > n$ the integral is zero.

$$2^\circ. \quad \int_{-1}^{1} \frac{x - \lambda}{x - 1}(P_k(x) - P_{k-1}(x))P_n^{(0,\beta)}(x)dx$$
$$= S_n^k - S_n^{k-1} - \frac{\lambda - 1}{k} \sum_{r=1}^{k} (2k - 2r + 1)S_n^{k-r} ,$$

$0 \le k \le n$, where

$$S_n^k = \frac{(-1)^{n-k}2n!(\beta)_{n-k}(\beta + n + 1)_k}{(n - k)!(n + k + 1)!} .$$

REFERENCES

[1] Maurice Parodi: A propos des polynômes de Legendre, *C. R. Acad. Sci. Paris, Series* A *et* B, **270** (1970), 1023–1025.

[2] Gabor Szegö: Orthogonal polynomials, *Colloquium Publications*, Vol. XXIII, *New York*, (1959).