OPERATIONAL REPRESENTATION FOR THE BESSEL POLYNOMIALS

M. K. DAS. (Calcutta)

1. In 1960, Carlitz [1] gave the following remarkable operational formula involving Laguerre polynomials:

$$(1.\ 1)\qquad \prod_{j=1}^{n} (xD - x + \alpha + j) = n! \sum_{p=0}^{n} \frac{x^p}{p!} L_{n-p}^{(\alpha+p)}(x)\, D^p,$$

where

$$(1.\ 2)\qquad L_n^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} e^x D^n (x^{\alpha+n} e^{-x})$$

Since then Chatterjea [2] engaged in deriving a similar operational formula connected with the Bessel polynomials. Having derived such a result he called the attention of Prof. Carliz, who asked Chatterjea to communicate the results to Prof. H.W. Gould, because at the same time Prof. H. W. Gould had been working in the same direction. Seeing the results Prof. H. W. Gould remarked „I have since been trying to extend it to the Bessel polynomials, . . . and I see that you have found the way to do it.“ Chatterjea proved the interesting result

$$(1.\ 3)\qquad \prod_{j=1}^{n} \{x^2 D + (2j + a)x + b\}$$

$$= \sum_{p=0}^{n} \binom{n}{p} b^{n-p} x^{2p} y_{n-p}(x, a + 2p + 2, b)\, D^p,$$

where $y_n(x, a, b)$ is defined by

$$(1.\ 4)\qquad y_n(x, a, b) = b^{-n} x^{2-a} e^{b/x} D^n (x^{2n+a-2} e^{-b/x})$$

Thus we have proved that

(2. 4) $$\cdot x^{-a} D^n (x^{2n+a} e^{-b/x} Y) = x^{-n} [x(\delta + a + 1)]^n (x^n e^{-b/x} Y)$$

Next we observe the following result of Chatterjea:

(2. 5) $$x^{-a} e^{b/x} D^n (x^{2n+a} e^{-b/x} Y) = \sum_{p=0}^{n} \binom{n}{p} b^{n-p} x^{2p} y_{n-p} (x, a + 2p + 2, b) D^p Y$$

It follows from (2. 4) and (2. 5) that

(2. 6) $$[x(\delta + a + 1)]^n (x^n e^{-b/x} Y) = e^{-b/x} \sum_{p=0}^{n} \binom{n}{p} b^{n-p} x^{n+2p} y_{n-p} (x, a + 2p + 2, b) D^p Y$$

which is (1. 6).

We shall now note some special case of (2. 6). When $Y = 1$ we have

$$[x(\delta + a + 1)]^n \cdot (x^n e^{-b/x}) = (bx)^n e^{-b/x} y_n (x, a + 2, b).$$

In other words, we get

(2. 7) $$[x(\delta + a - 1)]^n (x^n e^{-b/x}) = (bx)^n e^{-b/x} y_n (x, a, b).$$

Putting $a = b = 1$ in (2. 7) we derive

(2. 8) $$(x^2 D)^n \left(x^n e^{-\frac{1}{x}}\right) = x^n e^{-\frac{1}{x}} y_n (x, 1, 1)$$

Again using $a = b = 2$ in (2. 7) we obtain

(2. 9) $$[x(\delta + 1)]^n (x^n e^{-2/x}) = 2^n x^n e^{-2/x} y_n (x).$$

Next we proceed to make a generalisation of the result (2. 6). First we have

$$b^{-n} x^{-a} D^n (x^{2n+a} e^{-b/x} Y)$$

$$= b^{-n} (\delta + a + 1)(\delta + a + 2) \ldots \quad \ldots (\delta + a + n)(x^n e^{-b/x} Y)$$

$$= b^{-n} x^{-k} (\delta + a - k + 1)(\delta + a - k + 2) \ldots\ldots (\delta + a - k + n)(x^{n+k} e^{-b/x} Y)$$

$$= b^{-n} x^{-k} \cdot x^{-n} [x(\delta + a - k + 1)]^n (x^{n+k} e^{-b/x} Y).$$

Thus we have proved that

$$x^{-a} D^n (x^{2n+a} e^{-b/x} Y) = x^{-(n+k)} [x(\delta + a - k + 1)]^n (x^{n+k} e^{-b/x} Y). \tag{2. 10}$$

It therefore follows from (2. 5) and (2. 10) that

$$[x(\delta + a - k + 1)]^n (x^{n+k} e^{-b/x} Y) = e^{-b/x} \sum_{p=0}^{n} \binom{n}{p} b^{n-p} x^{n+2p+k} y_{n-p}(x, a + 2p + 2, b) D^p Y \tag{2. 11}$$

which is (1. 7).

It may be noted that on putting $k = 0$, (2. 11) reduces to (2. 6). Thus (2. 11) may be treated as a generalisation of (2. 6).

We shall now mention some particular cases of (2. 11). Using $k = a$, we derive from (2. 11)

$$[x(\delta + 1)]^n (x^{n+a} e^{-b/x} Y) = e^{-b/x} \sum_{p=0}^{n} \binom{n}{p} b^{n-p} x^{n+a+2p} y_{n-p}(x, a + 2p + 2, b) D^p Y \tag{2. 12}$$

Next putting $Y = 1$, in (2. 12) we obtain

$$[x(\delta + 1)]^n (x^{n+a} e^{-b/x}) = e^{-b/x} b^n x^{n+a} y_n (x, a + 2, b) \tag{2. 13}$$

Again using $a = 0$ and $b = 2$ in (2. 13) we get

$$[x(\delta + 1)]^n (x^n e^{-2/x}) = e^{-2/x} 2^n x^n y_n (x), \tag{2. 14}$$

which is as (2. 9). But (2. 14) is now derived from (2. 13). Thus (2. 13) may well be compared with (2. 7).

Lastly putting $\Phi^n = [x(\delta + 1)]^n$ in (2. 13), wc have

(2. 15) $$\Phi^n (x^{n+a} e^{-b/x}) = e^{-b/x} b^n x^{n+a} y_n (x, a + 2, b)$$

Further we notice from (2. 15) that

(2. 16) $$\Phi^m \left(e^{-b/x} x^{n+a} y_n (x, a + 2, b)\right) = b^{2n+m} e^{-b/x} x^{n+a} y_{n+m} (x, a + 2 - m, b)$$

Again from (2. 12) we derive

(2. 17) $$\Phi^m \left(e^{-b/x} x^{n+a} y_n (x, a + 2, b)\right) = e^{-b/x} \sum_{p=0}^{m} \binom{m}{p} b^{m-p} x^{m+a+2p} y_{m-p}(x, a + 2p + 2, b) D^p y_n (x, a + 2, b).$$

Now we know that

(2. 18) $$D^p y_n (x, a, b) = \binom{n}{p} p! (n + a - 1)_p b^{-p} y_{n-p}(x, a + 2p, b).$$

It therefore follows from (2. 16) and (2. 17) that

(2. 19) $$y_{n+m} (x, a, b) = \sum_{p=0}^{\min(m,n)} \binom{m}{p} \binom{n}{p} p! (n + m + a - 1)_p b^{-2n-2p} x^{m+2p-n} \times$$
$$\times y_{m-p}(x, a + m + 2p, b) y_{n-p}(x, a + m + 2p, b),$$

which may be compared with the following formula of Chatterjea:

(2. 20) $$y_{n+m} (x, a, b) = \sum_{p=0}^{\min(m,n)} \binom{m}{p} \binom{n}{p} p! (m + 2n + a - 1)_p \left(\frac{x}{b}\right)^{2p} \times$$
$$\times y_{n-p}(x, a + 2p, b) y_{m-p}(x, a + 2n + 2p, b).$$

I am indebted to Dr. S. K. Chatterjea for his kind help and guidance during the preparation of this note.

REFERENCES

[1] L. Carlitz, A note on the Laguerre polynomials, Michigan Math. Journal, Vol. 7 (1960), pp 219—223

[2[S. K. Chatterjea, Some problems connected with Special Functions, Thesis approved for Ph. D. degree, Jadavpur Univ. (India), 1964, Chapter IX.