GENERATING FUNCTIONS FOR POWERS OF CERTAIN FUNDAMENTAL SEQUENCES OF NUMBERS

MATHEMATICAL STRUCTURES — COMPUTATIONAL MATHEMATICS — MATHEMATICAL MODELLING
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Summary. Let $f(0), f(1), f(2), \ldots$ be a sequence defined by the equation $af(t+2)+bf(t+1)+cf(t)=0$ with $f(0)=\alpha$, $f(1)=\beta$ and a, b, c constants. Its generating function is $F_1(x)=\sum_{t=0}^{\infty} f(t)\,x^t$. We give the generating functions for powers of $f(t)$ expressing them in an explicit form as well the generating functions of $f(t)$ for the multiple argument.

1. Introduction. Let $f(0), f(1), f(2), \ldots$ be a sequence defined by the difference equation

$$af(t+2)+bf(t+1)+cf(t)=0, \tag{1}$$

with the initial values

$$f(0)=\alpha,\ f(1)=\beta.$$

Its generating function is

$$F_1(x)=\sum_{t=0}^{\infty} f(t)x^t=\frac{a\alpha+(a\beta+b\alpha)x}{a+bx+cx^2}.$$

More generally we put

$$F_k(x)=\sum_{t=0}^{\infty} f^k(t)x^t. \tag{2}$$

Riordan [1] has given a recurrence relation for $F_k(x)$ in the case $b=c=-a=-1$ and $\alpha=\beta=1$. Carlitz [2] generalized the relation of Riordan by different methods supplying new results. Recently, Horadam [3] has obtained analogous formulas for a certain generalized sequence of numbers. He considered also [4] generating functions of the form (2), when $f(t)$ satisfies a third order recurrence relation.

The purpose of this paper is to give new generating functions for powers of $f(t)$ expressing them in an explicit form. Generating functions for the function $f(t)$ of multiple argument will be added, too.

2. Fundamental sequences. The sequence $f(t)$ is the general solution of the equation (1), i. e.,

$$f(t)=C_1r_1^t+C_2r_2^t,$$

where r_1 and r_2 are the roots of the characteristic equation of (1) and C_1, C_2 constants determined by the initial values. We assume that $r_1\neq r_2$.

Not diminishing the generality of the problem, we consider the cases as follows:

1°. $\alpha=1$, $\beta=-\frac{b}{a}$; we have the sequence

$$f(t)=\frac{r_1^{t+1}-r_2^{t+1}}{r_1-r_2}, \tag{3}$$

and

2°. $\alpha=2$, $\beta=-\frac{b}{a}$, when we have got the companion sequence

$$\varphi(t)=r_1^t+r_2^t. \tag{4}$$

3. Properties of fundamental sequences. From (3) and (4) it follows that [5]

$$4r_i^{t+s+2}=\Delta f(t)f(s)+\varphi(t+1)\varphi(s+1)+(-1)^{i-1}\sqrt{\Delta}\,(f(t)\varphi(s+1)+f(s)\varphi(t+1)),$$

$$i=1,\,2,\ \sqrt{\Delta}=(r_1-r_2)a, s=0,\,1,\,2,\dots$$

Therefore

$$(5)\qquad \begin{aligned}2f(t+s+1)&=f(t)\varphi(s+1)+f(s)\varphi(t+1),\\ 2\varphi(t+s+2)&=\varphi(t+1)\varphi(s+1)+\Delta f(t)f(s).\end{aligned}$$

Since

$$f(-s-1)=-q^{-s}f(s-1),\quad \varphi(-s)=q^{-s}\varphi(s),$$

we find

$$(6)\qquad \begin{aligned}2q^{s+1}f(t-s-1)&=f(t)\varphi(s+1)-f(s)\varphi(t+1),\\ 2q^{s+1}\varphi(t-s)&=\varphi(t+1)\varphi(s+1)-\Delta f(t)f(s).\end{aligned}$$

Now we get from (5) and (6) that

$$\begin{aligned}f(t+s+1)&=f(t)\varphi(s+1)-q^{s+1}f(t-s-1),\\ \varphi(t+s+2)&=\varphi(t+1)\varphi(s+1)-q^{s+1}\varphi(t-s),\end{aligned}$$

which may be written as

$$(7)\qquad \begin{aligned}f((t+2)s-1)&=\varphi(s)f((t+1)s-1)-q^sf(ts-1)\\ \varphi(ts)&=\varphi(s)\varphi((t-1)s)-q^s\varphi((t-2)s).\end{aligned}$$

4. Generating functions for $f(t)$ and $\varphi(t)$ of multiple argument. The relations (7) above permits us to obtain generating functions for the functions $f(t)$ and $\varphi(t)$ when the argument is a multiple. Really, if we multiply each term of the relations (7) by x^t and sum from $t=0$ to $t=\infty$ we get

$$(8)\qquad (1-\varphi(s)x+q^sx^2)f_s(x)=f(s-1),$$

where

$$(9)\qquad f_s(x)=\sum_{t=0}^{\infty}f((t+1)s-1)x^t,$$

and

$$(10)\qquad (1-\varphi(s)x+q^sx^2)\varphi_s(s)=\varphi(s)-xq^s\varphi(0),$$

where

$$(11)\qquad \varphi_s(x)=\sum_{t=0}^{\infty}\varphi((t+1)s)x^t.$$

We conclude also that

$$(12)\qquad (1-\varphi(s)x+q^sx^2)\widetilde{\varphi}_s(x)=\varphi(0)-x\varphi(s),$$

where

$$(13)\qquad \widetilde{\varphi}_s(x)=\sum_{t=0}^{\infty}\varphi(ts)x^t.$$

5. Generating functions for powers of $f(t)$ and $\varphi(t)$ Following Lucas [5] the k-th powers of $f(t)$ and $\varphi(t)$ are

$$(14)\qquad \Delta^{[k/2]}f^k(t)=\sum_{r=0}^{[k/2]}(-1)^rC_k^rq^{r(t+1)}\psi_{k,r}(t),$$

with

$$\psi_{k,r}(t)=\begin{cases}f((k-2r)(t+1)-1), & k\text{—odd},\\ \widetilde{\varphi}((k-2r)(t+1)), & k\text{—even},\end{cases}$$

and

$$(15)\qquad \varphi^k(t)=\sum_{r=0}^{[k/2]}C_k^rq^{rt}\widetilde{\varphi}((k-2r)t),$$

with

$$\widetilde{\varphi}((k-2r)t)=\begin{cases}\varphi((k-2r)t), & k\neq 2r,\\ \frac{1}{2}\,\varphi((k-2r)t), & k=2r.\end{cases}$$

Multiplying the relation (14), respectively (15), by x^t and summing for $t=0, 1, 2, \ldots$ we obtain

$$\Delta^{[k/2]}F_k(x)=\sum_{r=0}^{[k/2]}(-1)^r C_k^r q^r \sum_{t=0}^{\infty}(xq^r)^t \psi_{k,r}(t),$$

and

$$\Phi_k(x)=\sum_{r=0}^{[k/2]} C_k^r \sum_{t=0}^{\infty}(xq^r)^t \widetilde{\varphi}((k-2r)t).$$

Taking into consideration (9) and (11) we find

(16)
$$\Delta^{[k/2]}F_k(x)=\sum_{r=0}^{[k/2]}(-1)^r C_k^r q^r \widetilde{\psi}_{k,r}(xq^r),$$

where

$$\widetilde{\psi}_{k,r}(xq^r)=\begin{cases} f_{k-2r}(xq^r), & k-\text{odd},\\ \widetilde{\varphi}_{k-2r}(xq^r), & k-\text{even}.\end{cases}$$

Similarly, for the function $\varphi(t)$ with (13) we have

(17)
$$\Phi_k(x)=\sum_{r=0}^{[k/2} C_k^r \widetilde{\varphi}_{k-2r}(xq^r).$$

If we replace $f_s(x)$, $\varphi_s(x)$ and $\widetilde{\varphi}_s(x)$ from (8), (10) and (12) in (16) and (17) they become

$$\Delta^{[k/2]}F_k(x)=\sum_{r=0}^{[k/2]}\frac{(-1)^r C_k^r q^r \theta_{k,r}(x)}{1-\varphi(k-2r)q^r x+q^k x^2},$$

with

$$\theta_{k,r}(x)=\begin{cases} f(k-2r-1), & k-\text{odd},\\ \varphi(k-2r)-xq^r\varphi(0), & k-\text{even}\neq 2r,\\ \widetilde{\varphi}(k-2r)-xq^r\widetilde{\varphi}(0), & k=2r\end{cases}$$

and

$$\Phi_k(x)=\sum_{r=0}^{[k/2]}\frac{C_k^r \chi_{k,r}(x)}{1-\varphi(k-2r)\,q^{rx}+q^k x^2}$$

with

$$\chi_{k,r}(x)=\begin{cases}\varphi(0)-xq^r\widetilde{\varphi}(k-2r), & k\neq 2r,\\ \widetilde{\varphi}(0)-xq^r\widetilde{\varphi}(k-2r), & k=2r.\end{cases}$$

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