# GENERATING FUNCTIONS FOR POWERS OF CERTAIN SECOND-ORDER RECURRENCE SEQUENCES

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## 1. INTRODUCTION

Let $u(n)$ and $v(n)$ be two sequences of numbers defined by

(1) $$u(n) = \frac{r_1^{n+1} - r_2^{n+1}}{r_1 - r_2}, \quad n = 0, 1, 2, \cdots$$

and

(2) $$v(n) = r_1^n + r_2^n, \quad n = 0, 1, 2, \cdots,$$

where $r_1$ and $r_2$ are the roots of the equation $ax^2 + bx + c = 0$.

It is known that the generating functions of these sequences are

$$u_1(x) = \left(1 + \frac{b}{a}x + \frac{c}{a}x^2\right)^{-1} \quad \text{and} \quad v_1(x) = \left(2 + \frac{b}{a}x\right)\left(1 + \frac{b}{a}x + \frac{c}{a}x^2\right)^{-1}.$$

We put

(3) $$u_k(x) = \sum_{n=0}^{\infty} u^k(n)x^n$$

and

(4) $$v_k(x) = \sum_{n=0}^{\infty} v^k(n)x^n.$$

J. Riordan [1] found a recurrence for $u_k(x)$ in the case $b = c = -a$. L. Carlitz [2] generalized the result of Riordan giving the recurrence relations for $u_k(x)$ and $v_k(x)$. A. Horadam [3] obtained a recurrence which unifies the preceding ones. He and A. G. Shannon [4] considered third-order recurrence sequences, too.

The object of this paper is to give the new recurrence relations for $u_k(x)$ and $v_k(x)$ such as the explicit form of the same generating functions. The generating functions of $u(n)$ and $v(n)$ for the multiple argument will be given, too. We use the result of E. Lucas [5].

## 2. RELATIONS OF $u(n)$ AND $v(n)$

From (1) and (2) we have

$$4r_i^{m+n+2} = \Delta u(n)u(m) + v(n+1)v(m+1) + (-1)^{i-1}\sqrt{\Delta}\,(u(n)v(m+1) + u(m)v(n+1)), \quad i = 1, 2,$$

with $\Delta = (b^2 - 4ac)/a^2$.

Then it follows that

$$2u(m+n+1) = u(n)v(m+1) + u(m)v(n+1)$$
$$2v(m+n+2) = v(n+1)v(m+1) + \Delta u(n)u(m).$$

Since

$$u(-n-1) = -q^{-n}u(n-1), \quad v(-n) = -q^{-n}v(n),$$

we find the relations

(5) $$u((n+2)m - 1) = u((n+1)m - 1)v(m) - q^m u(nm - 1),$$

(6) $$v(nm) = v((n-1)m)v(m) - q^m v((n-2)m).$$

From the identity

$$r_1^{kn} + r_2^{kn} = \sum_{r=0}^{[k/2]} (-1)^r \frac{k}{k-r} C_{k-r}^r (r_1^n + r_2^n)^{k-2r}(r_1 r_2)^{rn},$$

if we put $u(n)$ and $v(n)$ we get

(7) $$v(kn) = \sum_{r=0}^{[k/2]} (-1)^r \frac{k}{k-r} C_{k-r}^r q^{rn} v^{k-2r}(n), \quad k \geqslant 1.$$

Similarly, from

$$2r_i^{n+1} = v(n+1) + (-1)^{i-1}\sqrt{\Delta}\,u(n), \quad i = 1, 2,$$

and taking into consideration

$$\sum_{s=0} \binom{p+s}{s}\binom{2p+m}{2p+2s} = 2^{m-1}\,\frac{2p+m}{m}\binom{m+p-1}{p},$$

we obtain

(8)
$$\sum_{r=0}^{[k/2]} \Delta^{[k/2]-r}\,\frac{k}{k-r}\,C_{k-r}^{r}\,q^{r(n+1)}u^{k-2r}(n) = \lambda_k(n),$$

where

$$\lambda_k(n) = \begin{cases} u(k(n+1)-1), & k \text{ odd},\\ v(k(n+1)), & k \text{ even}.\end{cases}$$

## 3. GENERATING FUNCTIONS OF $u(n)$ AND $v(n)$ FOR MULTIPLE ARGUMENT

The relations (5) and (6) give us the possibility to find the generating functions of $u(n)$ and $v(n)$ when the argument is a multiple. Indeed, we obtain from (5)

(9)
$$(1 - v(m)x + q^m x^2)u(m,x) = u(m-1),$$

where

(10)
$$u(m,x) = \sum_{n=0}^{\infty} u((n+1)m-1)x^n.$$

From (6) we have

(11)
$$(1 - v(m)x + q^m x^2)v(m,x) = v(m) - q^m v(0)x,$$

where

(12)
$$v(m,x) = \sum_{n=0}^{\infty} v((n+1)m)x^n .$$

We find also

(13)
$$(1 - v(m)x + q^m x^2)\tilde{v}(m,x) = v(0) - v(m)x,$$

with

$$\tilde{v}(m,x) = v(0) + v(m,x)x.$$

## 4. RECURRENCE RELATIONS OF $u_k(x)$ AND $v_k(x)$

Let us now return to (8) and consider the sum

$$\sum_{r=0}^{[k/2]} \Delta^{[k/2]-r}\,\frac{k}{k-r}\,C_{k-r}^{r}q^r \sum_{n=0}^{\infty} u^{k-2r}(n)(q^r x)^n = \sum_{n=0}^{\infty} \lambda_k(n)x^n$$

which by (3), (10) and (12) yields the following relation

$$\Delta^{[k/2]}u_k(x) = \lambda(k,x) - \sum_{r=1}^{[k/2]} \Delta^{[k/2]-r}\,\frac{k}{k-r}\,C_{k-r}^{r}q^r u_{k-2r}(q^r x),$$

where

$$\lambda(k,x) = \begin{cases} u(k,x), & k \text{ odd}\\ v(k,x), & k \text{ even}.\end{cases}$$

Similarly from (7) for $v_k(x)$ follows

$$v_k(x) = \tilde{v}(k,x) + \sum_{r=1}^{[k/2]} (-1)^{r-1}\,\frac{k}{k-r}\,C_{k-r}^{r}v_{k-2r}(q^r x).$$

## 5. EXPLICIT FORM OF $u_k(x)$ AND $v_k(x)$

Next we construct the powers for $u(n)$ and $v(n)$. From (1) and (2) we obtain

(14)
$$\Delta^{[k/2]}u^k(n) = \sum_{r=0}^{[k/2]} (-1)^r C_k^r q^{r(n+1)}\lambda_{k-2r}(n),$$

and

$$(15) \qquad v^k(n) = \sum_{r=0}^{[k/2]} C_k^r q^{rn} \tilde{v}((k-2r)n),$$

where

$$\tilde{v}(t) = \begin{cases} v(t), & t \neq 0, \\ \tfrac{1}{2}v(t), & t = 0. \end{cases}$$

Hence we multiply each member of the equations (14) and (15) by $x^n$ and sum from $n = 0$ to $n = \infty$. By (3) and (4) the following generating functions for powers of $u(n)$ and $v(n)$ are obtained:

$$\Delta^{[k/2]} u_k(x) = \sum_{r=0}^{[k/2]} (-1)^r C_k^r q^r \lambda(k-2r, q^r x),$$

and

$$v_k(x) = \sum_{r=0}^{[k/2]} C_k^r v(k-2r, q^r x).$$

If we replace $u(m,x)$, $v(m,x)$ and $\tilde{v}(m,x)$ from (9), (11) and (13), we get

$$\Delta^{[k/2]} u_k(x) = \sum_{r=0}^{[k/2]} \frac{(-1)^r C_k^r q^r \mu_{kr}(x)}{1 - v(k-2r)q^r x + q^k x^2},$$

where

$$\mu_{kr} = \begin{cases} u(k-2r-1), & k \text{ odd}, \\ v(k-2r) - q^r v(0)x, & k \text{ even}, k \neq 2r \\ \tilde{v}(k-2r) - q^r v(0)x, & k = 2r, \end{cases}$$

and

$$v_k(x) = \sum_{r=0}^{[k/2]} \frac{C_k^r \omega_{kr}(x)}{1 - v(k-2r)q^r x + q^k x^2},$$

where

$$\omega_{kr} = \begin{matrix} v(0) - q^r v(k-2r)x, & k \neq 2r, \\ \tilde{v}(0) - q^r \tilde{v}(k-2r)x, & k = 2r. \end{matrix}$$

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