ЗА НЕКОИ ИНТЕГРАЛИ КОИ ВЛЕГУВААТ ВО ИЗУЧУВАЊЕТО НА ПОЛИНОМИТЕ НА JACOBI И НА LAGUERRE

Прилози МАНУ, Оддел. за мат.-тех. науки, I/1, 1980, 5-17

1. Да ги разгледаме полиномите $P_n^{(\alpha,\beta)}(x)$ на Jacobi определени со рекурентната формула [1]

$$2n(n+\alpha+\beta)(2n+\alpha+\beta-2)\,P_n^{(\alpha,\beta)}(x)$$

$$=(2n+\alpha+\beta-1)\{(2n+\alpha+\beta)(2n+\alpha+$$

$$+\beta-2)x+\alpha^2-\beta^2\}P_{n-1}^{(\alpha,\beta)}(x)-2(n+\alpha-1)(n+$$

$$+\beta-1)(2n+\alpha+\beta)P_{n-2}^{(\alpha,\beta)}(x),\qquad n=2,3,4,\ldots$$

$$P_0^{(\alpha,\beta)}(x)=1,\quad P_1^{(\alpha,\beta)}(x)=\frac{1}{2}(\alpha+\beta+2)x-\frac{1}{2}(\alpha-\beta).$$

Познато е, дека тие образуваат една база на векторскиот простор на полиномите од степен најмногу еднаков на n и дека се ортогонални на $[-1,1]$ со тежинска функција $(1-x)^\alpha(1+x)^\beta$. Притоа имаме $\alpha,\beta>-1$.

Методата на Christoffel-Darboux за овие полиноми нѐ води до следниов идентитет

$$\text{(1)}\quad \sum_{k=0}^{n}\{h_k^{(\alpha,\beta)}\}^{-1}P_k^{(\alpha,\beta)}(x)\,P_k^{(\alpha,\beta)}(y)=\frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}$$

$$\cdot\frac{\Gamma(n+2)\,\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}$$

$$\cdot\frac{P_{n+1}^{(\alpha,\beta)}(x)\,P_n^{(\alpha,\beta)}(y)-P_n^{(\alpha,\beta)}(x)\,P_{n+1}^{(\alpha,\beta)}(y)}{x-y}$$

каде што е

$$h_k^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}}{2k+\alpha+\beta+1}\;\frac{\Gamma(k+\alpha+1)\,\Gamma(k+\beta+1)}{\Gamma(k+1)\,\Gamma(k+\alpha+\beta+1)}.$$

Нека $f(x)$ е произволен полином од степен n, определен со

$$\text{(2)}\qquad f(x)=\sum_{k=0}^{n}a_k\,P_k^{(\alpha,\beta)}(x).$$

Од (1) ќе имаме

$$h_n^{(\alpha,\beta)}\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta f(x)\sum_{k=0}^{n}\{h_k^{(\alpha,\beta)}\}^{-1}P_k^{(\alpha,\beta)}(x)\,P_k^{(\alpha,\beta)}(y)\,dx$$

$$= \frac{2(n+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)_2} \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)$$

$$\cdot \frac{P_{n+1}(x) P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x) P_{n+1}^{(\alpha,\beta)}(y)}{x-y} dx$$

$$(\alpha)_n = \prod_{k=1}^{n} (\alpha+k-1), \quad (\alpha) = 1, \quad n \geqslant 1.$$

По замена во левата страна на последново равенство со вредноста на $f(x)$ од (2) добиваме

$$\frac{(2n+\alpha+\beta+1)_2 h_n^{(\alpha,\beta)}}{2(n+1)(n+\alpha+\beta+1)} \sum_{k=0}^{n} a_k P_k^{(\alpha,\beta)}(y)$$

$$= \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) \frac{P_{n+1}^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x) P_{n+1}^{(\alpha,\beta)}(y)}{x-y} dx$$

или

$$(3) \qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) \frac{P_{n+1}^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(y) - P_n^{(\alpha,\beta)}(x) P_{n+1}^{(\alpha,\beta)}(y)}{x-y} dx =$$

$$= \frac{(2n+\alpha+\beta+1)_2 h_n^{(\alpha,\beta)}}{2(n+1)(n+\alpha+\beta+1)} f(y).$$

Последнава релација овозможува да се пресметаат бројни интеграли.

2. Така, поради

$$P_n^{(\alpha,\beta)}(1) = \binom{\alpha+n}{n} = \frac{(\alpha+1)_n}{n!},$$

земајќи $y = 1$, од (3) ќе имаме

$$(4) \qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) \frac{(n+1) P_{n+1}^{(\alpha,\beta)}(x) - (n+\alpha+1) P_n^{(\alpha,\beta)}(x)}{x-1} dx$$

$$= \frac{n!\,(2n+\alpha+\beta+1)_2 h_n^{(\alpha,\beta)}}{2(\alpha+1)_n (n+\alpha+\beta+1)} f(1).$$

Имајќи предвид дека е

$$\frac{(n+1) P_{n+1}^{(\alpha,\beta)}(x) - (n+\alpha+1) P_n^{(\alpha,\beta)}(x)}{x-1} = \frac{2n+\alpha+\beta+2}{2} P_n^{(\alpha+1,\beta)}(x),$$

од (4) следува

$$(5) \qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x) P_n^{(\alpha+1,\beta)}(x)\, dx$$

$$= \frac{(2n+\alpha+\beta+1)\cdot n!\ h_n^{(\alpha,\beta)}}{(\alpha+1)_n\,(n+\alpha+\beta+1)}\, f(1)$$

$$= 2^{\alpha+\beta+1}\, B(n+\beta+1,\ \alpha+1)\, f(1),$$

каде што е

$$B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,.$$

Слично поради

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{\beta+n}{n} = (-1)^n \frac{(\beta+1)_n}{n!}\,,$$

од (3) за $y = -1$, ќе имаме

$$(6)\qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\, \frac{(n+1)\,P_n^{(\alpha,\beta)}(x) + (n+\beta+1)\,P_n^{(\alpha,\beta)}(x)}{x+1}\, dx$$

$$= \frac{(-1)^n\, n!\,(2n+\alpha+\beta+1)_2\, h_n^{(\alpha,\beta)}}{2\,(n+\alpha+\beta+1)\,(\beta+1)_n}\, f(-1).$$

Бидејќи е

$$\frac{(n+1)\,P_n^{(\alpha,\beta)}(x) + (n+\beta+1)\,P_n^{(\alpha,\beta)}(x)}{x+1} = \frac{2n+\alpha+\beta+2}{2}\, P_n^{(\alpha,\beta+1)}(x)$$

од (6) се добива

$$(7)\qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta f(x)\, P_n^{(\alpha,\beta+1)}(x)\, dx$$

$$= \frac{(-1)^n\, n!\,(2n+\alpha+\beta+1)}{(n+\alpha+\beta+1)\,(\beta+1)_n}\, f(-1)$$

$$= (-1)^n\, 2^{\alpha+\beta+1}\, B(n+\alpha+1,\ \beta+1)\, f(-1).$$

Земајќи во предвид дека е [2]

$$P_n^{(\alpha,\beta-1)}(x) - P_n^{(\alpha-1,\beta)}(x) = P_{n-1}^{(\alpha,\beta)}(x)$$

од (5) и (7) наоѓаме

$$(8)\qquad \int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta-1} f(x)\, P_n^{(\alpha,\beta)}(x)\, dx$$

$$= 2^{\alpha+\beta-1}\,[B(n+\beta+1,\ \alpha)\, f(1) + (-1)^n B(n+\alpha+1,\ \beta)\, f(-1)].$$

3. Да разгледаме некои посебни случаи.

Нека е

$$(9)\qquad f(x) = P_n^{(\gamma,\delta)}(x), \quad \gamma, \delta > -1.$$

Од (5) добиваме

$$\int_{-1}^{1} (1-x)^\alpha (1+x)^\beta P_n^{(\alpha+1,\beta)}(x)\, P_n^{(\gamma,\delta)}(x)\, dx$$

$$= \frac{(2n+\alpha+\beta+1)\,(\gamma+1)_n}{(n+\alpha+\beta+1)\,(\alpha+1)_n}\, h_n^{(\alpha,\beta)}\,,$$

додека од (7) ќе имаме

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-1}P_n^{(\alpha,\beta)}(x)\,P_n^{(\gamma,\delta)}(x)\,dx$$
$$=\frac{(2n+\alpha+\beta)\,(\delta+1)_n}{(n+\alpha+\beta)\,(\beta)_n}\;h_n^{(\alpha,\beta-1)}.$$

Ако ја внесеме вредноста за $f(x)$ од (9) во (8) наоѓаме

(10) $$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta-1}P_n^{(\alpha,\beta)}(x)P_n^{(\gamma,\delta)}(x)\,dx$$
$$=2^{\alpha+\beta-1}\left[\binom{n+\gamma}{n}B(n+\beta+1,\,\alpha)+\binom{n+\delta}{n}B(n+\alpha+1,\,\beta)\right]$$
$$=\frac{(2n+\alpha+\beta+1)\,h_n^{(\alpha,\beta)}}{n+\alpha+\beta+1}\left(\frac{(\gamma+1)_n}{(\alpha+1)_n}+\frac{(\delta+1)_n}{(\beta+1)}\right).$$

4. Ако земеме

$$f(x)=x^n,$$

од (5) добиваме

$$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta}\,x^n P_n^{(\alpha,\beta)}(x)\,dx=2^{\alpha+\beta}B(n+\beta+1,\,\alpha)$$

и слично од (7)

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta-1}\,x^n P_n^{(\alpha,\beta)}dx=2^{\alpha+\beta}B(n+\alpha+1,\,\beta).$$

Од (8) во овој случај наоѓаме

$$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta-1}x^n P_n^{(\alpha,\beta)}(x)\,dx$$
$$=2^{\alpha+\beta-1}\left[B(n+\alpha+1,\,\beta)+B(n+\beta+1,\,1)\right]$$
$$=2^{\alpha+\beta-1}B(\alpha,\beta)\frac{(\alpha)_{n+1}+(\beta)_{n+1}}{(\alpha+\beta)_{n+1}}.$$

5. Да земеме

$$f(x)=P_n(x),$$

каде што $P_n(x)$ е полином на Legendre определен со

$$P_n(x)=\sum_{k=0}^{\left[\frac{n}{2}\right]}\frac{(-1)^k\left(\frac{1}{2}\right)_{n-k}(2x)^{n-2k}}{k!\,(n-2k)!}.$$

Знаеме дека е

$$P_n(1) = (-1)^n P_n(-1) = 1,$$

па имаме

$$\int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta-1} P_n(x) P_n^{(\alpha, \beta)}(x)\, dx$$

$$= \int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta-1} x^n P_n^{(\alpha, \rho)} dx$$

$$= 2^{\alpha+\beta-1} (B(n+\alpha+1, \beta) + B(n+\beta+1, \alpha)).$$

што се направо добива и од (10) зимајки $\gamma = \delta = 0$ т. е. за

$$P_n(x) = P_n^{(0, 0)}(x).$$

6. Нека е

$$f(x) = L_n^{(\alpha)}(x),$$

каде што $L_n^{(\alpha)}(x)$ е полином на Laguerre определен со

(11) $$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (-x)^k}{k!\,(n-k)!\,(1+\alpha)_k}.$$

Познато е дека

$$L_n^{(\alpha)}(x) = \frac{(\alpha+1)_n}{n!} \; {}_1F_1(-n, 1+\alpha, x),$$

каде што е

$${}_1F_1(-n; 1+\beta; x) = \sum_{k=0}^{n} \frac{(-n)_k x^k}{(1+\alpha)_k}.$$

Од (5) имаме

$$\int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta} L_n^{(\alpha)}(x) P_n^{(\alpha, \beta)}(x)\, dx$$

$$= 2^{\alpha+\beta} \frac{(1+a)_n}{n!} B(n+\beta+1, n-\alpha) \; {}_1F_1(-n; 1+a; 1),$$

додека од (8) наоѓаме

$$\int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta-1} L_n^{(\alpha)}(x) P_n^{(\alpha, \beta)}(x)\, dx$$

$$= 2^{\alpha+\beta-1} [B(n+\beta+1, \alpha) \; {}_1F_1(-n; 1+a; 1) +$$

$$+ (-1)^n B(n+\alpha+1, \beta) \; {}_1F_1(-n, 1+a, -1)] \frac{(1+a)_n}{n!}.$$

7. Да земеме

$$f(x) = H_n(x),$$

каде што $H_n(x)$ е полином на Hermite определен со

$$H_n(x) = \sum_{k=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^k\, n!\; (2x)^{n-2k}}{k!\,(n-2k)!} .$$

Од (5) наоѓаме

$$\int_{-1}^{1} (1-x)^{\alpha-1}(1+x)^{\beta} H_n(x) P_n^{(\alpha,\beta)}(x)\, dx = 2^{\alpha+\beta+1} B(n+\beta+1, \alpha) H_n(1),$$

а од (8) имаме

$$\int_{-1}^{1} (1-x)^{\alpha-1}(1+x)^{\beta-1} H_n(x)\; P_n^{(\alpha,\beta)}(x)\; dx$$

$$= 2^{\alpha+\beta-1}\left[B(n+\beta+1, \alpha)\; H_n(1) + (-1)\; B(n+\alpha+1, \beta)\; H_n(-1)\right].$$

8. Полиномите на Laguerre определени со (11) ја задоволуваат следнава рекурентна формула

$$n L_n^{(\alpha)}(x) = (-x + 2n + \alpha - 1)\; L_{n-1}^{(\alpha)} - (n + \alpha - 1)\; L_{n-2}^{(\alpha)}(x)$$

$$L_0^{(\alpha)}(x) = 1, \quad L_1^{(\alpha)}(x) = -x + \alpha + 1.$$

Методот на Christoffel-Darboux за овие полиноми ни дава

$$(12) \quad \sum_{k=0}^{n} \frac{k!\; L_k^{(\alpha)}(x)\, L_k^{(\alpha)}(y)}{(1+\alpha)_k} = \frac{(n+1)!}{(1+\alpha)_n}\; \frac{L_{n+1}^{(\alpha)}(x)\, L_n^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(y)\; L_n^{(\alpha)}(x)}{y-x} .$$

Да земеме еден полином од обликот

$$f(x) = \sum_{s=0}^{n} a_s L_s^{(\alpha)}(x).$$

Од (12) добиваме

$$\sum_{k=0}^{n} \frac{k!}{(1+\alpha)_k} L_k^{(\alpha)}(y) \int_0^{\infty} e^{-x} x^{\alpha}\; L_k^{(\alpha)}(x) \sum_{s=0}^{n} a_s L_p^{(\alpha)}(s)\, dx$$

$$= \frac{(n+1)!}{(1+\alpha)_n} \int_0^{\infty} s^{-x} x^{\alpha} f(x)\; \frac{L_{n+1}^{(\alpha)}(x)\, L_n^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(y)\, L_n^{(\alpha)}(x)}{y-x}\; dx$$

или

$$\int_0^{\infty} e^{-x} x^{\alpha} f(x)\; \frac{L_{n+1}^{(\alpha)}(x) L_n^{(\alpha)}(y) - L_{n+1}^{(\alpha)}(y) L_n^{(\alpha)}(x)}{y-x}\; dx = \frac{\Gamma(n+\alpha+1)}{(n+1)!} f(y).$$

9. Во посебниот случај, кога е $y = 0$, поради

$$L_n^{(\alpha)}(0) = \frac{(1+\alpha)_n}{n!}$$

добиваме

$$(13)\qquad \int_0^\infty e^{-x} x^\alpha f(x) \frac{(n+\alpha+1) L_n^{(\alpha)}(x) - (n+1) L_{n+1}^{(\alpha)}(x)}{x}\, dx = \Gamma(\alpha+1) f(0).$$

Бидејќи е

$$x L_n^{(\alpha+1)}(x) = (n+\alpha+1)\, L_n^{(\alpha)}(x) - (n+1)\, L_{n+1}^{(\alpha)}(x),$$

од (13) добиваме

$$(14)\qquad \int_0^\infty x^\alpha e^{-x} f(x)\, L_n^{(\alpha+1)}(x)\, dx = \Gamma(\alpha+1) f(0).$$

Ако е

$$f(x) = x^n,$$

го добиваме познатото својство дека е

$$\int_0^\infty e^{-x} x^{\alpha+1} L_n^{(\alpha+1)}(x)\; x^{n-1}\, dx = 0.$$

Да земеме

$$f(x) = L_n^{(\beta)}(x).$$

Од (14) ќе имаме

$$(15)\qquad \int_0^\infty e^{-x} x^\alpha L_n^{(\alpha+1)}(x)\, L_n^{(\beta)}(x)\, dx = \frac{(1+\beta)_n}{n!}\, \Gamma(\alpha+1).$$

Ако е $\beta = \alpha$, поради

$$L_k^{(\alpha+1)}(x) = \sum_{k=0}^{n} L_k^{(\alpha)}(x),$$

од (15) ќе имаме

$$\int_0^\infty e^{-x} x^\alpha\, L_n^{(\alpha+1)}(x)\, L_n^{(\alpha)}(x)\, dx = \frac{\mathrm{F}(n+\alpha+1)}{n!}\,.$$

10. Нека е

$$f(x) = P_n(x).$$

Од (14) добиваме

$$\int_0^\infty e^{-x} x^\alpha L_n^{(\alpha+1)}(x)\; P_n(x)\, dx = L(\alpha+1)\, {}_2F_1\left(-n, n+1;\, 1, \frac{1}{2}\right).$$

Но бидејќи е

$$F\left(-2n,\, 2n+1;\, 1;\, \frac{1}{2}\right) = \frac{(-1)^n \left(\frac{1}{2}\right)_n}{n!}$$

и

$$F\left(-2n+1,\; 2n+2;\, 1,\, \frac{1}{2}\right) = 0,$$

ќе имаме

$$\int_{0}^{\infty} e^{-x} x^{\alpha} P_{2n}(x)\ L_{2n}^{(\alpha+1)}(x)\, dx = \frac{(-1)^n \left(\frac{1}{2}\right)_n}{n!} \Gamma(\alpha+1)$$

или ако $\alpha = n$, добиваме

$$\int_{-0}^{\infty} e^{-x} x^{n} L_{2n}^{(n+1)}(x)\ P_{2n}(x)\, dx = (-1)^n \left(\frac{1}{2}\right)_n .$$

Слично, ако земеме

$$f(x) = H_{2n}(x),$$

поради

$$H_{2n}(0) = (-1)^n\, 2^{2n} \left(\frac{1}{2}\right)_n, \quad H_{2n+1}(0) = 0,$$

ќе имаме

$$\int_{0}^{\infty} e^{-x} x^{\alpha} H_{2n}(x)\ L_{2n}^{(\alpha+1)}(x)\ dx = (-1)^n\, 2^n (2n-1)!!\ \Gamma(\alpha+1),$$

или ако е $\alpha = n$

$$\int_{0}^{\infty} e^{-x} x^{\alpha}\ H_{2n}(x)\ L_{2n}^{(n+1)}(x)\, dx = (-1)^n (2n)!.$$

11. Познато е дека

$$L_n^{(\alpha+\beta+1)}(x) = \sum_{k=0}^{n} L_k^{(\alpha)}(x)\, L_{n-k}^{(\beta)}(x).$$

Во тој случај од (14) ќе имаме

$$\int_{0}^{\infty} e^{-x} x^{\alpha-1} L_n^{(\alpha)}(x)\, L_n^{(\alpha+\beta)}(x+y)\, dx = \Gamma(\alpha) \sum_{k=0}^{n} \frac{(\alpha)_k}{k!} L_{n-k}^{(\beta)}(y).$$

Но бидејќи е

$$\sum_{k=0}^{n} \frac{(\alpha)_k}{k!}\ L_{n-k}^{(\beta)}(y) = L_n^{(\alpha+\beta)}(y),$$

од (14) ќе добиеме

$$\int_{0}^{\infty} e^{-x} x^{\alpha-1} L_n^{(\alpha)}(x)\, L_n^{(\alpha+\beta)}(y+x)\, dx = \Gamma(\alpha)\, L_n^{(\alpha,\beta)}(y).$$

Слично од

$$L_n^{(\alpha)}(xy) = \sum_{k=0}^{n} \frac{(1+\alpha)_n (1-y)^{n-k} y^k L_k^{(\alpha)}(x)}{(n-k)!\ (1+\alpha)_k},$$

следува

$$\int_{0}^{\infty} e^{-x} x^{\alpha} L_n^{(\alpha+1)}(x)\, L_n^{(\beta)}(xy)\, dx = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}\ \frac{\Gamma(\beta+n+1)}{\Gamma(n+1)} .$$

12. Земајќи во предвид дека е

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(\alpha - \beta)_k \, L_{n-k}^{(\beta)}(x)}{k!},$$

за произволни α и β, и од (14) ќе имаме

$$\int_0^{\infty} e^{-x} x^{\alpha} L_n^{(\alpha+1)}(x) L_n^{(\beta)}(x)\, dx = \frac{(1+\beta)_n}{n!} \Gamma(\alpha+1)$$

$$= \sum_{k=0}^{n} \frac{(\beta-\alpha)_k}{k!} \frac{\Gamma(\alpha+n-k+1)}{(n-k)!},$$

од каде следува идентитетот

$$\sum_{k=0}^{n} \frac{(\beta-\alpha)_k (\alpha+1)_{n-k}}{k!\,(n-k)!} = \frac{(1+\beta)_n}{n!}.$$

13. За полиномите на Hermite $H_n(x)$, ја имаме формулата на Christoffel-Darboux

$$2 \sum_{k=0}^{n} \frac{1}{2^k k!} H_k(x) H_k(y) = \frac{1}{2^n n!} \frac{H_{n+1}(x) H_n(y) - H_n(x) H_{n+1}(y)}{x-y}.$$

Ако $f(x)$ е полином од степен n, т.е.

$$f(x) = \sum_{s=0}^{n} a_s H_s(x),$$

се добива

$$2\sqrt{\pi} \sum_{k=0}^{n} a_k H_k(y) = 2\sqrt{\pi} f(y)$$

$$= \frac{1}{2^n n!} \int_{-\infty}^{\infty} e^{-x^2} f(x) \frac{H_{n+1}(x) H_n(y) - H_n(x) H_{n+1}(y)}{x-y}\, dx.$$

Земајќи $y = 0$, добиваме

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{H_{2n}(x)}{x} f(x)\, dx = (-1)^n \frac{2^{n+1}(2n)!}{(2n-1)!!} f(0),$$

каде што $f(x)$ е полином од степен $2n$.

Ако земеме

$$f(x) = L_{2n}^{(\alpha)}(x),$$

ќе добиеме

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{H_{2n}(x)}{x} L_{2n}^{(\alpha)}(x) = (-1)^n 2^{2n+1} (1+\alpha)_n.$$

Слично имаме

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{H_{2n}(x)}{x} P_{2n}(x)\, dx = 2^{n+1}(2n-1)!!$$

или поопшто

$$\int_{-\infty}^{\infty} e^{-x^2} \frac{H_{2n}(x)}{x} C_{2n}^{\nu}(x)\, dx = 2^{2n+1} (\nu)_n,$$

каде што $C_n^{\nu}(x)$ е полином на Gegenbauer, определен со

$$C_n^{\gamma}(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (\nu)_{n-k} (2x)^{n-2k}}{k!\ (n-2k)!} .$$

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SUMMARY

ON SOMME INTEGRALS INVOLVING IN STUDIES OF JACOBI AND LAGUERRE POLYNOMIALS

The following formulas are established:

1°. $$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x)\, P_n^{(\alpha+1,\,\beta)}(x)\, dx = 2^{\alpha+\beta+1} B(n+\beta+1, \alpha+1) f(1)$$

2°. $$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} f(x)\, P_n^{(\alpha,\,\beta+1)}(x) dx = (-1)^n\, 2^{\alpha+\beta+1} B(n+\alpha+1, \beta+1) f(-1)$$

3°. $$\int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta-1} L_n^{(a)}(x)\, P_n^{(\alpha,\,\beta)}\, dx =$$
$$= 2^{\alpha+\beta+1} [B(n+\beta+1, \alpha)\, {}_1F_1(-n; 1+a; 1) +$$
$$+ (-1)^n B(n+\alpha+1, \beta)\, {}_1F_1(-n. 1+a, -1)] \frac{(1+a)_n}{n!}$$

4°. $$\int_{-1}^{1} (1-x)^{\alpha-1} (1+x)^{\beta-1} H_n(x)\, P_n^{(\alpha,\,\beta)}(x)\, dx =$$
$$= 2^{\alpha+\beta-1} [B(n+\beta+1, \alpha) H_n(1) + (-1)^n B(n+\alpha+1, \beta) H_n(-1)]$$

5°. $$\int_{0}^{\infty} e^{-x} x^{\alpha} L_n^{(\alpha+1)}(x) P_n(x)\, dx = \Gamma(\alpha+1)\, {}_2F_1\left(-n;\ n+1;\ \frac{1}{2}\right)$$

6°. $$\int_{0}^{\infty} e^{-x}\ x^{\alpha} H_{2n}(x) L_{2n}^{(\alpha+1)}(x)\, dx = (-1)^n 2^n (2n-1)!!\, \Gamma(\alpha+1)$$

where $P_n^{(\alpha,\,\beta)}(x)$, $L_n^{(a)}(x)$, $H_n(x)$ are Jacobi, Laguerre, Hermite polynomials respectively and $f(x) = \sum_{k=0}^{n} G_k P_k^{(\alpha,\,\beta)}(x)$.