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THE CONSTRUCTION OF A CORP IN THE SET OF POINTS IN A LINE OF DESARGUES AFFINE PLANE

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Abstract. In the article [1], we show that the set of points on a line, in the affine Desargues plans, connected with addition forms an Abelian group. In this article, we will define multiplication of points on a line in the affine Desargues plans. We will show that this set forms a multiplicative group. And we will show that every straight line of Desargues affine plans, along with both addition and multiplication operations, forms the corp (skew-field).

1. Introduction, Desargues affine plane, commutative group $(OI, +)$

Definition 1. [3, 10, 11] Affine plane is called the *incidence structure* $\mathcal{A} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ that satisfies the following axioms:

Axiom 1. For every two different points P and $Q \in \mathcal{P}$, there exists exactly one line $\ell \in \mathcal{L}$ incident with that points.

The line ℓ, determined from the point P and Q will be denoted by PQ.

Axiom 2. For a point $P \in \mathcal{P}$, and an line $\ell \in \mathcal{L}$ such that $(P, \ell) \notin \mathcal{I}$, there exists one and only one line $r \in \mathcal{L}$ incident with the point P and such that $\ell \cap r = \varnothing$.

Axiom 3. In $\mathcal{A}$ there are three non-incident points with a line.

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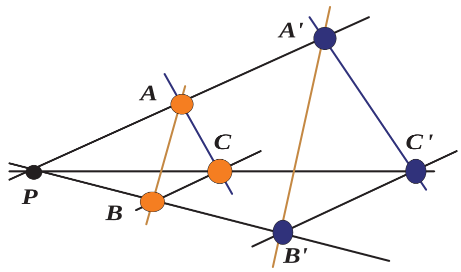

FIGURE 1.

The fact $(P, \ell) \in \mathcal{I}$, (equivalent to $P\mathcal{I}\ell$) we mark $P \in \ell$ and read *point* P *is incident with a line* ℓ or a line passes through points P (contains point P). Whereas a line of the affine plane we consider as sets of points of affine plane with her incidents. Axiom 1 implicates that tow different lines of $\mathcal{L}$ many have a common point, in other words tow different lines of $\mathcal{L}$ either have no common point or have only one common point.

Definition 2. Two lines $\ell, m \in \mathcal{L}$ that are matching or do not have common point are called *parallel* and in this case we write $\ell \parallel m$; when they have only one common point we say that they are expected.

For a single line $r \in \mathcal{L}$, which passes through a point $P \in \mathcal{P}$ and is parallel with line AB, that does not pass through the point P, we will use the notation ℓ^P_{AB}.

Proposition 1.1. [4, 10, 12, 13] *Parallelism relation* $\parallel = \{(r, s) \in \mathcal{L}^2 \mid r \parallel s\}$ *on* $\mathcal{L}$ *is an equivalence relation in* $\mathcal{L}$.

Definition 3. Three different points $P, Q, R \in \mathcal{P}$ are called *collinear*, if there is incidence with the same straight line.

Definition 4. The set of three different non-collinear points A, B, C together with the line AB, BC, CA is called *three-vertex* and is marked as ABC.

Proposition 1.2. [6, 7, 9, 10, 22] (The Desargues affine plane theorem). *If* $ABC, A'B'C'$ *are two three-vertex but not with the same vertices in an affine plane (Fig. 1), then*

$$\begin{matrix} AC \parallel A'C' \\ BC \parallel B'C' \end{matrix} \Longrightarrow AB \parallel A'B'$$

In affine Euclidean plane this proposition holds

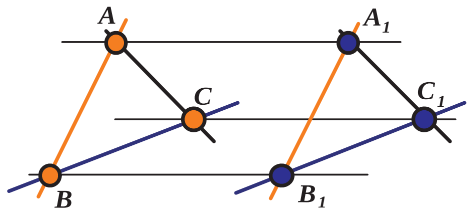

Figure 2.

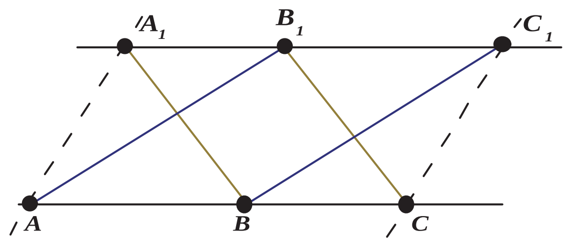

Figure 3.

Proposition 1.3. (Axiom I of Desargues) *If AA_1, BB_1, CC_1 are three different parallel lines (Fig. 2), then*

$$\begin{matrix} AB \parallel A_1B_1 \\ BC \parallel B_1C_1 \end{matrix} \Longrightarrow AC \parallel A_1C_1$$

There are affine plans where Proposition 1.3 is not valid. Such is the Moulton plane [10].

Definition 5. [2, 7, 10] An affine plane complete with Desargues axiom (Proposition 1.3), is called *Desargues affine plane.*

Let A, B, C be three different points of a line and A_1, B_1, C_1 three different points of another parallel to the first (Fig.3). If $AB_1 \parallel BC_1$ and $A_1B \parallel B_1C$ is $AA_1 \parallel CC_1$? Otherwise, we add the problem if we have this

Proposition 1.4. [1, 17, 18] ("Little Pappus Theorem"). *Let A, B, C and A_1, B_1, C_1 be two triple points located in two parallel lines (Fig. 3). If $AB_1 \parallel BC_1$ and $BA_1 \parallel CB_1$ then $AA_1 \parallel CC_1$ holds.*

Theorem 1. [1, 17] ("Little Hessenberg Theorem") *For a Desargues plane Propositions 1.4 is true.*

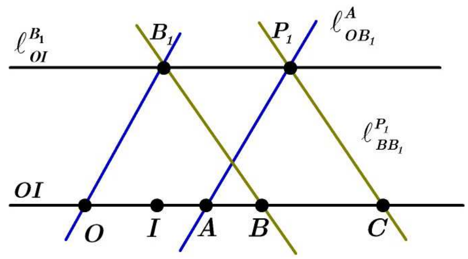

FIGURE 4.

In an Desargues affine plane $\mathcal{D} = (\mathcal{P}, \mathcal{L}, \mathcal{I})$ we fix two different points $O, I \in \mathcal{P}$, which, according to Axiom 1, determine a line $OI \in \mathcal{L}$. Let A and B be two arbitrary points of a line OI. In plane $\mathcal{D}$ we choose a point B_1 not incident with OI: $B_1 \notin OI$ (we call the auxiliary point). Construct line $\ell_{OI}^{B_1}$, which is only according to the Axiom 2. Then construct line $\ell_{OB_1}^{A}$, which also is the only according to the Axiom 2. Marking their intersection $P_1 = \ell_{OI}^{B_1} \cap \ell_{OB_1}^{A}$. Finally construct line $\ell_{BB_1}^{P_1}$. For as much as BB_1 expects OI in point B, then this line, parallel with BB_1, expects line OI in a single point C (Fig.4).

The process of construct the points C, starting from two whatsoever points A, B of the line OI, is presented in the algorithm form

Algorithm 1.

$$\begin{aligned}&\textbf{Step.1}.\ B_1 \notin OI\\ &\textbf{Step.2}.\ \ell_{OI}^{B_1} \cap \ell_{OB_1}^{A} = P_1\\ &\textbf{Step.3}.\ \ell_{BB_1}^{P_1} \cap OI = C\end{aligned}$$

The point C is determined in single mode (does not depend on the choice auxiliary point B_1) by Algorithm 1 [1] .

Definition 6. [1] In the above conditions, operation

$$+ : OI \times OI \longrightarrow OI$$

defined by $(A, B) \mapsto C$ for all $(A, B) \in OI \times OI$ we call the addition in OI.

According to this definition, one can write

$$(\forall A, B \in OI) \begin{array}{l}\textbf{Step.1}.\ B_1 \notin OI\\ \textbf{Step.2}.\ \ell_{OI}^{B_1} \cap \ell_{OB_1}^{A} = P_1\\ \textbf{Step.3}.\ \ell_{BB_1}^{P_1} \cap OI = C\end{array} \Leftrightarrow A + B = C.$$

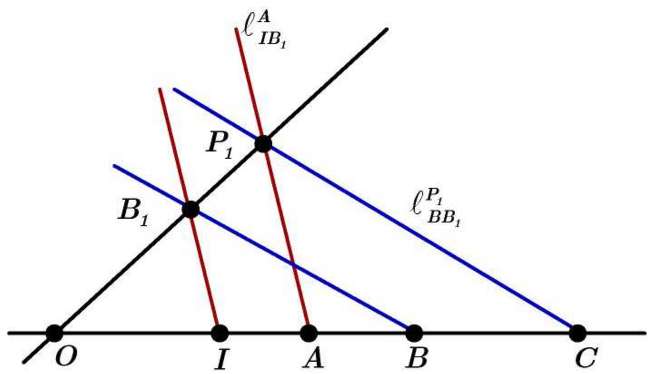

FIGURE 5.

Theorem 2. [1] *The groupoid* $(OI, +)$ *is commutative (Abelian) group; the zero element is the point* O*.*

2. Multiplication of points on a line in Desargues affine plane and its properties

Choose in the plane $\mathcal{D}$ one point B_1 not incident with lines OI, which together with point I forming the line IB_1. Construct the line $\ell^{A}_{IB_1}$, which is the only according to the Axiom 2 and cutting the line OB_1. Marking their intersection with $P_1 = \ell^{A}_{IB_1} \cap OB_1$. Finally, construct the line $\ell^{P_1}_{BB_1}$. Since BB_1 meets the line OI in point B, then this line, parallel with BB_1, meets the line OI in one single point C (Fig.5).

The process of construct the points C, is presented in the algorithm form

Algorithm 2.

Step.1. $B_1 \notin OI$

Step.2. $\ell^{A}_{IB_1} \cap OB_1 = P_1$

Step.3. $\ell^{P_1}_{BB_1} \cap OI = C$

In the process of construct the points C, except pairs (A, B) of points $A, B \in OI$, is required and the selection of point $B_1 \notin OI$, which we call the auxiliary point to point C. The following theorem demonstrates that the choice of auxiliary point does not affect the position of point C in line OI, determined by the Algorithm 2.

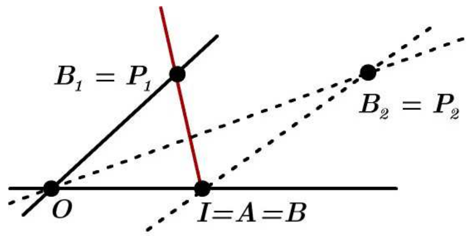

FIGURE 6.

Theorem 3. *For every two points $A, B \in OI$, the Algorithm 2 determines a single point $C \in OI$, which does not depend on the choice of its auxiliary point B_1.*

Proof. According to the Algorithm 2, by selecting the point $B_1 \notin OI$ for a given pair of points (A, B) of the line OI, construct the point C. Now we choose another point B_2. Then, according to Algorithm 2, construct analog the point C', that in these conditions is found as:

$$\left[\begin{array}{l} \textbf{Step.1}.\ B_2 \notin OI \\ \textbf{Step.2}.\ \ell_{OI}^{B_2} \cap \ell_{OB_2}^{A} = P_2 \\ \textbf{Step.3}.\ \ell_{BB_2}^{P_2} \cap OI = C' \end{array}\right], \tag{2.1}$$

We distinguish these four cases of the position of points A, B in relation to fixed point I of the line OI.

Case 1. $A = B = I$. By the choice of the point B_1, according to Algorithm 2, have:

$$P_1 = \ell_{IB_1}^{I} \cap OB_1 = B_1 \Longrightarrow C = \ell_{BB_1}^{B_1} \cap OI = IB_1 \cap OI = I;$$

From the choice of the point B_2, according to (2.1) have:

$$P_2 = \ell_{IB_2}^{I} \cap OB_2 = B_2 \Longrightarrow C' = \ell_{IB_2}^{B_1} \cap OI = IB_2 \cap OI = I.$$

Therefore accept the $C = C' = I$ (Fig.6).

Case 2. $A = I \neq B$. By the choice of the point B_1 have

$$P_1 = \ell_{IB_1}^{I} \cap OB_1 = B_1 \Longrightarrow C = \ell_{BB_1}^{B_1} \cap OI = BB_1 \cap OI = B;$$

From the choice of the point B_2 have

$$P_2 = \ell_{IB_2}^{I} \cap OB_2 = B_2 \Longrightarrow C = \ell_{BB_2}^{B_2} \cap OI = BB_2 \cap OI = B.$$

Therefore in this case accept the $C = C' = B$ (Fig.7).

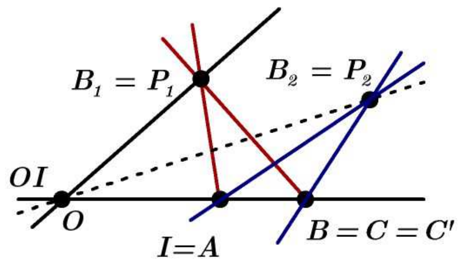

FIGURE 7.

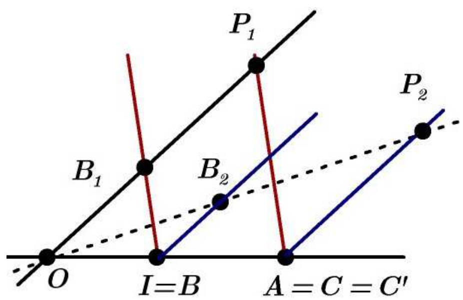

FIGURE 8.

Case 3. $A \neq I = B$. The situation is analogous to the second case, where point B takes the role of point A and conversely, so in this case we have $C = C' = A$ (Fig.8).

Case 4. $A \neq B \neq I$. Here we distinguish two sub-cases.

a) In the case where points I, B_1, B_2 are collinear points, by the choice of the point B_1 have

$$P_1 = \ell^{A}_{IB_1} \cap OB_1 \Longrightarrow C = \ell^{P_1}_{BB_1} \cap OI;$$

from the choice of the point B_2 have

$$P_2 = \ell^{A}_{IB_2} \cap OB_2 \Longrightarrow C' = \ell^{P_2}_{BB_2} \cap OI.$$

From Algorithm 2 and (2.1) appears also that, collinearity of points I, B_1, B_2 induce collinearity of the points A, P_1, P_2.

Examine three-vertices BB_1B_2 and CP_1P_2 (Fig.9). We note that $B_1B_2 || P_1P_2$. But $C \in \ell^{P_1}_{BB_1} \parallel BB_1$ therefore $BB_1 || CP_1$.

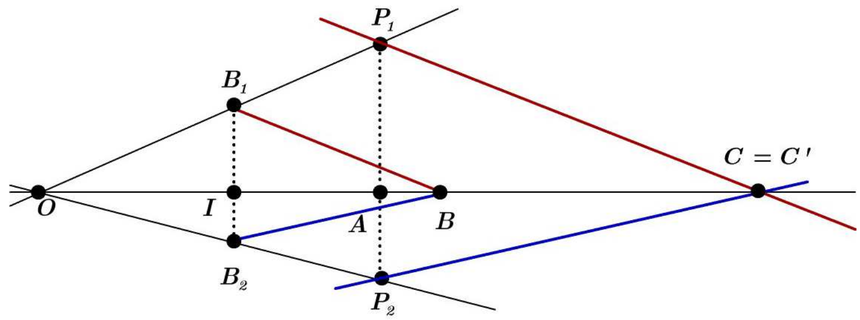

FIGURE 9.

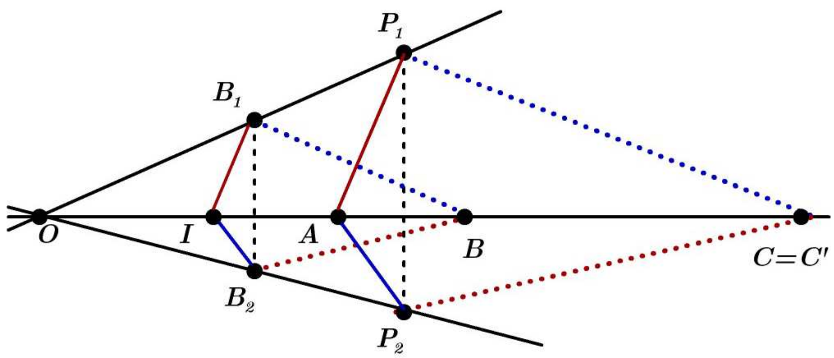

FIGURE 10.

From here, the Desargues affine plane theorem (Proposition 1.2), results that $B_2B||P_2C$. On the other hand, $C' \in \ell_{BB_2}^{P_2} \Longrightarrow P_2C' \parallel B_2B$, which is parallel to P_2C. Consequently $C' \in P_2C$, which means that $C = C'$.

b) The points I, B_1, B_2 are non-collinear. Here we distinguish two subcases related to fixed point O:

$\mathbf{b}_1$) The points O, B_1, B_2 are non-collinear (Figure 10);

$\mathbf{b}_2$) The points O, B_1, B_2 are collinear (Figure 11).

In case $\mathbf{b_1}$), from the choice of point B_1, have: $P_1 = \ell_{IB_1}^{A} \cap OB_1 \Longrightarrow C = \ell_{BB_1}^{P_1} \cap OI$;

from the choice of point B_2, according to (2.1) have: $P_2 = \ell_{IB_2}^{A} \cap OB_2 \Longrightarrow C' = \ell_{BB_2}^{P_2} \cap OI$.

From Algorithm 2, and (2.1) we get also that, non-collinearity of points I, B_1, B_2 delivers non-collinearity of points A, P_1, P_2 .

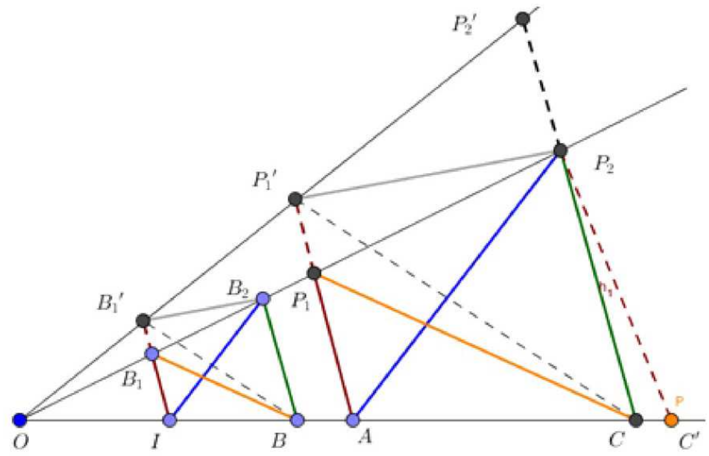

FIGURE 11.

We consider three-vertices IB_1B_2 and AP_1P_2. By Desargues affine plane theorem, we obtain that $B_1B_2 \parallel P_1P_2$.

Now consider three-vertices BB_1B_2 and CP_1P_2. Again by Desargues affine plane theorem, we take $B_2B \parallel P_2C$. On the other hand, C', that delivers $P_2C' \parallel B_2B$. Consequently $C' \in P_2C$, which means that $C = C'$.

In the case $\mathbf{b_2}$), again we have

$$P_1 = \ell^{A}_{IB_1} \cap OB_1 \Longrightarrow C = \ell^{P_1}_{BB_1} \cap OI; P_2 = \ell^{A}_{IB_2} \cap OB_2 \Longrightarrow C' = \ell^{P_2}_{BB_2} \cap OI.$$

In the line CP_2 we take another point P_2' and construct the line OP_2'. Mark $B_1' = OP_2' \cap IB_1$ and $P_1' = OP_2' \cap AP_1$. We examine three-vertices $IB_1'B_2$ and $AP_1'P_2$. We have: $IB_1' \parallel AP_1', IB_2 \parallel AP_2$, therefore, by Desargues affine plane theorem, we take from $B_1'B_2 \parallel P_1'P_2$. Now examine three-vertices BB_1B_1' and CP_1P_1'; by Desargues affine plane theorem, we take from $BB_1' \parallel CP_1'$. Finally we examine three-vertices $BB_1'B_2$ and $CP_1'P_2$. We have: $BB_1' \parallel CP_1', B_1'B_2 \parallel P_1'P_2$, therefore we take from $BB_2 \parallel CP_2$. But $BB_2 \parallel C\prime P_2$, and therefore $C = C'$.

□

Let A and B be two arbitrary points of the line OI. We associate pairs $(A, B) \in OI \times OI$ point $C \in OI$, that determine algorithm Algorithm 2.

According to the preceding Theorems, point C is determined in single mode. Thus we obtain an application $OI \times OI \longrightarrow OI$.

Definition 7. In the above conditions, we call the operation

$$* : OI \times OI \longrightarrow OI,$$

defined by $(A, B) \mapsto C$ for all $(A, B) \in OI \times OI$, multiplication in OI.

According to this definition, one can write

$$(\forall A, B \in OI,) \left[\begin{array}{l} \textbf{Step1}.\ B_1 \notin OI, \\ \textbf{Step2}.\ \ell^{A}_{IB_1} \cap OB_1 = P_1 \\ \textbf{Step3}.\ \ell^{P_1}_{BB_1} \cap OI = C. \end{array} \right] \iff A * B = C. \quad (2.2)$$

From here, the following proposition is obvious.

$$(\forall A \in OI)\, O * A = A * O = O. \quad (2.3)$$

3. Properties of the multiplication in the line OI

By Theorem 3, this is immediately true.

Proposition 3.1. *Multiplication $*$ in OI has identity element the point I:*

$$(\forall A \in OI)\, I * A = A * I = I. \quad (3.1)$$

The following propositions are also valid.

Proposition 3.2. *The multiplication $*$ is associative in OI :*

$$(\forall A, B, D \in OI)\ \ (A * B) * D = A * (B * D)\,. \quad (3.2)$$

Proof. In the case where at least one of the points A, B, D is point O, from (2.3), equation (3.2) is evident, whereas in the case where at least one of the points A, B, D is point I, it comes from (3.1). We eliminate in the case where $A, B, D \neq O; A, B, D \neq I$ and $A \neq B \neq D$ (when at least two points are the same, equally justify).

Firstly we construct the product $(A * B) * D$. In this case (Fig.12), according to (2.2), for $A*B$, connected to auxiliary point for multiplication we have

$$\left\{ \begin{array}{l} \textbf{1}.\ \ B_1 \notin OI, \\ \textbf{2}.\ \ell^{A}_{IB_1} \cap OB_1 = P_1, \\ \textbf{3}.\ \ell^{P_1}_{BB_1} \cap OI = C. \end{array} \right\} \Longrightarrow A * B = \ell^{P_1}_{BB_1} \cap OI \quad (3.3)$$

$$\Longrightarrow \left\{ \begin{array}{c} IB_1 \parallel AP_1 \\ BB_1 \parallel (A * B)\, P_1 \end{array} \right\}$$

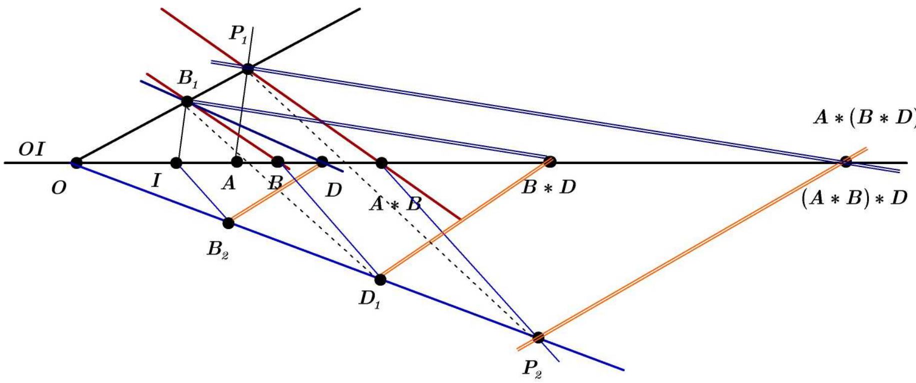

FIGURE 12.

Choose the point B_2 (Fig.12) as auxiliary points for the construction of multiplication $(A * B) * D$.

Construct the line $\ell_{IB_2}^{A*B}$ and mark $P_2 = \ell_{IB_2}^{A*B} \cap OB_2$. Then, according to (2.2) have

$$\left\{ \begin{array}{l} \mathbf{1}.\ B_2 \notin OI, \\ \mathbf{2}.\ OB_2 \cap \ell_{IB_2}^{A*B} = P_2, \\ \mathbf{3}.\ \ell_{DB_2}^{P_2} \cap OI = C. \end{array} \right\} \Longrightarrow (A * B) * D = \ell_{DB_2}^{P_2} \cap OI \tag{3.4}$$

$$\Longrightarrow \left\{ \begin{array}{c} IB_2 \parallel (A * B)\, P_2 \\ DB_2 \parallel [(A * B) * D]\, P_2 \end{array} \right\}$$

Now construct multiplication $A * (B * D)$. Choose as the auxiliary point for multiplication $B * D$ the point B_2. Construct the line $\ell_{IB_2}^{B}$ and mark $D_1 = \ell_{IB_2}^{B} \cap OB_2$. Then, according to (2.2) have

$$B * D = \ell_{DB_2}^{D_1} \cap OI \Longrightarrow \left\{ \begin{array}{c} IB_2 \parallel BD_1 \\ DB_2 \parallel [B * D]\, D_1. \end{array} \right\} \tag{3.5}$$

Choose the point B_1 (Fig.12) as auxiliary points for the construction of multiplication $A * (B * D)$. Construct the line $\ell_{(B*D)B1}^{P_1}$. Then, according to (2) have:

$$\left.\begin{array}{l}\textbf{1}.\ B_1 \notin OI,\\ \textbf{2}.\ OB_2 \cap \ell^{A}_{IB_1} = P_1,\\ \textbf{3}.\ \ell^{P_1}_{(B*D)B_1} \cap OI = C.\end{array}\right. \Longrightarrow A*(B*D) = \ell^{P_1}_{(B*D)B_1} \cap OI \tag{3.6}$$

$$\Longrightarrow \left\{ \begin{array}{c} IB_1 \parallel AP_1 \\ (B*D)\,B_1 \parallel [A*(B*D)]\,P_1 \end{array} \right\}$$

Whereas, the (3.4) and (3.5), have

$$\left\{ \begin{array}{c} BD_1 \parallel (A*B)\,P_2 \\ (B*D)\,D_1 \parallel [(A*B)*D]\,P_2 \end{array} \right\} \tag{3.7}$$

We examine three-vertices BB_1D_1 and $(A*B)P_1P_2$, to which, by (3.3) have $BB_1 \parallel (A*B)P_1$ and from (3.7) we have $BD_1 \parallel (A*B)P_2$. Therefore, the Desargues affine plane theorem , have $B_1D_1 \parallel P_1P_2$. We examine further three-vertices $(B*D)B_1D_1$ and $[(A*B)*D]P_1P_2$, for which, from above we have $B_1D_1 \parallel P_1P_2$ and from (3.7) we have $(B*D)D_1 \parallel [(A*B)*D]P_2$. Therefore we take from $(B*D)B_1 \parallel [(A*B)*D]P_1$. But by (3.6) we have also $(B*D)B_1 \parallel [A*(B*D)]P_1$, that brings $[(A*B)*D]P_1 \parallel [A*(B*D)]P_1$, and since the points $(A*B)*D$, $A*(B*D) \in OI$, we take $(A*B)*D = A*(B*D)$. □

Proposition 3.3. *For every point except O in OI, there exists its right symmetrical according to multiplication:*

$$(\forall A \in OI - \{O\})(\exists A^{-1} \in OI - \{O\})\, A*A^{-1} = I$$

Proof. We distinguish two cases: $A = I$ and $A \neq I, O$.

Case 1. If $A = I$, then $A^{-1} = I$ because, according to (3.1), $I*I = I$.

Case 2. If $A \neq I, O$, requested points $A^{-1} \in OI$, such that

1. $A_1^{-1} \notin OI$,

2. $\ell^{A}_{IA_1^{-1}} \cap OA_1^{-1} = P_1$,

3. $\ell^{P_1}_{A^{-1}A_1^{-1}} \cap OI = I$.

Given this, we take initially a point $A_1^{-1} \notin OI$ and construct the line IA_1^{-1}, and then the line $\ell^{A}_{IA_1^{-1}}$. Mark $P_1 = \ell^{A}_{IA_1^{-1}} \cap OA_1^{-1}$. Furthermore construct the line IP_1 and parallel with it by the points A_1^{-1} construct the line $\ell^{A_1^{-1}}_{IP_1}$. The latter is not parallel with the line OI, therefore expects that at some point: $\ell^{A_1^{-1}}_{IP_1} \cap OI \neq O$. It is clear that this point is the point A^{-1}

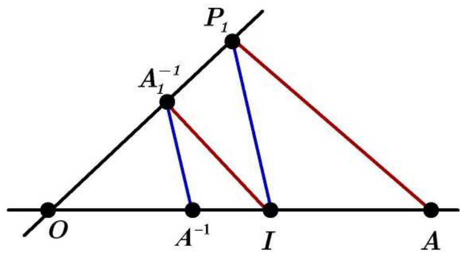

FIGURE 13.

(Fig.13), such that the $A * A^{-1} = I$ and $A^{-1} \neq O$. So the A^{-1} thus the resulting the right identity element of point A.

□

Due to the definition of group, [4], from Propositions 3.1, 3.2 and 3.3 we obtain this

Theorem 4. *In an Desargues affine plane the groupoid $(OI, *)$ is a group; identity element is the point I.*

Based on one theorem of algebra, right neutral element of an element of one group is neutral element of that element, [4],[5]. Therefore,

$$(\forall A \in OI - \{O\})(\exists A^{-1} \in OI - \{O\})\, A * A^{-1} = A^{-1} * A = I \qquad (3.8)$$

4. The algebra $(OI, +, *)$ is a corp in Desargues affine plane

Proposition 4.1. *The multiplication $*$ is distributive related to the addition $+$ in the line OI:*

$$\begin{aligned} &(i)\ \ (A + B) * D = A * D + B * D \\ &(ii)\ A * (B + D) = A * B + A * D \end{aligned} \qquad (4.1)$$

for every $A, B, D \in OI$.

Proof. (i) In the case where at least one of the points A, B, D is the point O , by the (2.3), equivalence (i) is evident. We eliminate in the case where $A, B, D \neq O$ and $A \neq B \neq D$ (when at least two points are the same, equally justify). We distinguish two sub-cases: $a)$ at least one of the points A, B, D is the point I ; $b)$ $A, B, D \neq I$.

$a)$ When $D = I$, according to (3.1), equalization (i) is evident. Let it be now $A = I$ (the case $B = I$ behaves in case $A = I$, based on a

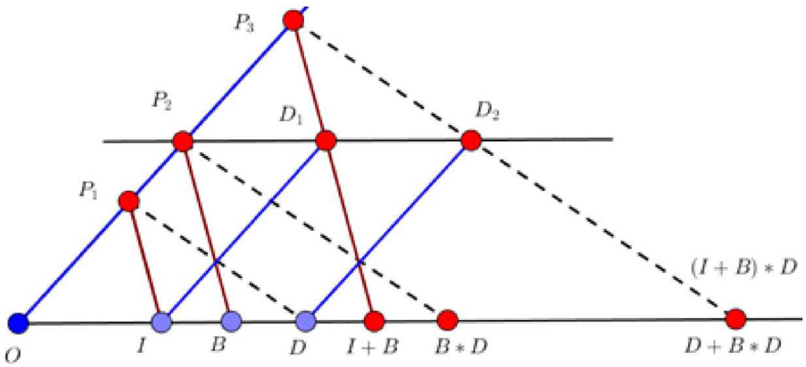

FIGURE 14.

commutative addition property in OI). For as much as $A \neq B \neq D$, have $B \neq I$ and $D \neq I$ (Fig.14). Equalisation (i), in this case takes the view $(I + B) * D = D + B * D$.

For the construction of multiplication $(I + B) * D$, construct firstly multiplication $B * D$, taking its auxiliary point the point $P_1 \notin OI$. Mark $P_2 = OP_1 \cap \ell^{B}_{IP_1}$. Then, according to the algorithm Algorithm 2, where in the role of A is B, in the role of B is D, in the role of B_1 is P_1, in the role of P_1 is P_2, have

$$\left.\begin{cases} \mathbf{1}.\ P_1 \notin OI, \\ \mathbf{2}.\ OP_1 \cap \ell^{B}_{IP_1} = P_2, \\ \mathbf{3}.\ \ell^{P_2}_{DP_1} \cap OI = C. \end{cases}\right\} \Longrightarrow B * D = \ell^{P_2}_{DP_1} \cap OI \tag{4.2}$$
$$\Longrightarrow \left\{ \begin{array}{c} IP_1 \parallel BP_2 \\ DP_1 \parallel (B * D)\, P_2 \end{array} \right\}$$

Construct further the sum $I + B$, by taking its auxiliaries point the point $P_2 \notin OI$. Mark $D_1 = \ell^{P_2}_{OI} \cap \ell^{I}_{OP_2}$. Then, according to the Algorithm 1, where in the role of A is I, in the role of B_1 is P_1, in the role of P_1 is P_2, have

$$\left.\begin{cases} \mathbf{1}.\ P_2 \notin OI, \\ \mathbf{2}.\ \ell^{P_2}_{OI} \cap \ell^{I}_{OP_2} = D_1, \\ \mathbf{3}.\ \ell^{D_1}_{BP_2} \cap OI = C. \end{cases}\right\} \Longrightarrow I + B = \ell^{D_1}_{BP_2} \cap OI \tag{4.3}$$
$$\Longrightarrow \left\{ \begin{array}{c} OP_2 \parallel ID_1 \\ BP_2 \parallel (I + B)\, D_1. \end{array} \right\}$$

Finally construct multiplication $(I+B)*D$, by taking its auxiliaries point the point $P_1 \notin OI$. Mark $P_3 = OP_1 \cap \ell_{IP_1}^{(I+B)}$. Then, according to the Algorithm 2, where in the role of A is $I+B$, in the role of B is D, in the role of B_1 is P_1 , in the role of P_1 is P_3, have

$$\begin{array}{l} \mathbf{1}.P_1 \notin OI, \\ \mathbf{2}.\ OP_1 \cap \ell_{IP_1}^{(I+B)} = P_3, \ \Longrightarrow (I+B)*D = \ell_{DP_1}^{P_3} \cap OI \\ \mathbf{3}.\ \ell_{DP_1}^{P_3} \cap OI = C. \\ \qquad \Longrightarrow \left\{ \begin{array}{c} IP_1 \parallel (I+B)\,P_3 \\ DP_1 \parallel [(I+B)*D]\,P_3. \end{array} \right\} \end{array} \tag{4.4}$$

Now construct the right side $D + B * D$ of equivalence, by taking as the auxiliaries point of sum the point $P_2 \notin OI$. Mark $D_2 = \ell_{OI}^{P_2} \cap \ell_{OP_2}^{D}$. Then, according to the Algorithm 1, where in the role of A is D, in the role of B is $B*D$, in the role of B_1 is P_2, in the role of P_1 is D_2, have:

$$\begin{array}{l} \mathbf{1}.\ P_2 \notin OI, \\ \mathbf{2}.\ \ell_{OI}^{P_2} \cap \ell_{OP_2}^{D} = D_2, \qquad \Longrightarrow D + (B*D) = \ell_{(B*D)P_2}^{D_2} \cap OI \\ \mathbf{3}.\ \ell_{(B*D)P_2}^{D_2} \cap OI = C. \\ \qquad \Longrightarrow \left\{ \begin{array}{c} OP_2 \parallel DD_2 \\ (B*D)\,P_2 \parallel [D+(B*D)]\,D_2 \end{array} \right\} \end{array} \tag{4.5}$$

By (4.2), (4.3) and (4.4) we have that $IP_1 \parallel BP_2 \parallel (I+B)D_1 \parallel (I+B)P_3$, which indicates that the points $(I+B)$, D_1 and P_3 are collinear.

We note that three-vertices IDP_1 and $D_1D_2P_3$ have respective vertices in parallel lines $ID_1 \parallel P_1P_2 \parallel DD_2$ and satisfy the Desargues affine plane theorem (DAPT) conditions, therefore

$$\begin{array}{c} IP_1 \parallel D_1P_3 \\ ID \parallel D_1D_2 \end{array} \overset{(DAPT)}{\Longrightarrow} ID_1 \parallel D_2P_3. \tag{4.6}$$

But, by (4.2) and (4.5), we have $DP_1 \parallel [D+B*D]\,D_2$. Since the parallelism is equivalence relation (Proposition 1.1), by the (4.6), we have $D_2P_3 \parallel [D+(B*D)]\,D_2$. So the, points P_3, D_2 and $[D+(B*D]$ are collineary. By (4.4) have $DP_1 \parallel [(I+B)*D]P_3$, that brings $[D+(B*D)]P_3 \parallel [(I+B)*D]P_3$. Consequently the resulting true equalization

$$(I+B)*D = D+B*D$$

b) $A, B, D \neq I$, where $A \neq B \neq D$

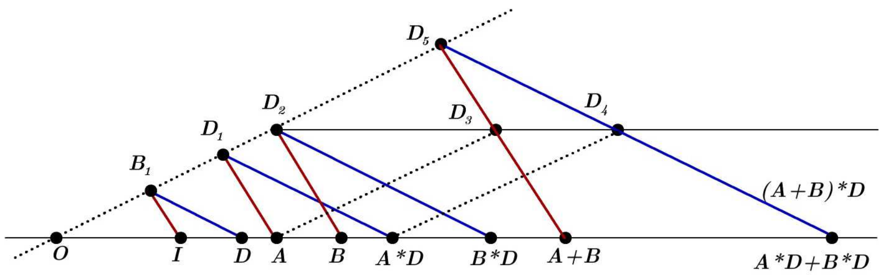

FIGURE 15.

To the construction of multiplication $(A+B)*D$, initially construct $A*D$ and $B*D$, (Fig.15). To multiplication $A*D$, according to Algorithm 2, where in the role of B is D, in the role of P_1 is D_1, have

$$\begin{array}{l} \textbf{1}.\ B_1 \notin OI, \\ \textbf{2}.\ OB_1 \cap \ell^A_{IB_1} = D_1, \\ \textbf{3}.\ \ell^{D_1}_{DB_1} \cap OI = C. \end{array} \Longrightarrow A*D = \ell^{D_1}_{DB_1} \cap OI. \tag{4.7}$$

Whereas for production $B*D$ have

$$\begin{array}{l} \textbf{1}.\ B_1 \notin OI, \\ \textbf{2}.\ OB_1 \cap \ell^B_{IB_1} = D_2, \\ \textbf{3}.\ \ell^{D_2}_{DB_1} \cap OI = C. \end{array} \Longrightarrow B*D = \ell^{D_2}_{DB_1} \cap OI. \tag{4.8}$$

From (4.7) and (4.8), have

$$\begin{gathered} IB_1 \parallel AD_1 \parallel BD_2 \\ DB_1 \parallel (A*D)\,D_1 \parallel (B*D)\,D_2 \end{gathered} \tag{4.9}$$

Construct further the sum $A+B$, by taking as its auxiliaries point, the point $D_2 \in OB_1$. Mark $D_3 = \ell^{D_2}_{OI} \cap \ell^A_{OD_2}$. Then, according to the Algorithm 1, where in the role of B_1 is D_2, in the role of P_1 is D_3, have

$$\begin{gathered} \left\{ \begin{array}{l} \textbf{1}.\ D_2 \notin OI, \\ \textbf{2}.\ \ell^{D_2}_{OI} \cap \ell^A_{OD_2} = D_3, \\ \textbf{3}.\ \ell^{D_3}_{D_2B} \cap OI = C. \end{array} \right\} \Longrightarrow A+B = \ell^{D_3}_{D_2B} \cap OI \\ \Longrightarrow \left\{ \begin{array}{c} D_2D_3 \parallel OI \\ AD_3 \parallel OD_2 \\ BD_2 \parallel (A+B)\,D_3 \end{array} \right\} \end{gathered} \tag{4.10}$$

Finally construct production $(A+B)*D$, by taking as its auxiliaries point, the point $B_1 \notin OI$. Mark $D_5 = OB_1 \cap \ell_{IB_1}^{(A+B)}$. Then, according to the Algorithm 2, where in the role of A is $A+B$, in the role of B is D, in the role of P_1 is D_5, have

$$\begin{array}{l} \mathbf{1}.\ B_1 \notin OI, \\ \mathbf{2}.\ OB_1 \cap \ell_{IB_1}^{(A+B)} = D_5, \quad \Longrightarrow (A+B)*D = \ell_{DB_1}^{D_5} \cap OI \\ \mathbf{3}.\ \ell_{DB_1}^{D_5} \cap OI = C. \\ \qquad \Longrightarrow \left\{ \begin{array}{c} IB_1 \parallel (A+B)\, D_5 \\ DB_1 \parallel [(A+B)*D]\, D_5 \end{array} \right\} \end{array} \tag{4.11}$$

Now construct the right-hand of the equalization *(i)* $A*D+B*D$, by taking as auxiliaries point of the sum the point $D_2 \notin OI$. Mark $D_4 = \ell_{OI}^{D_2} \cap \ell_{OD_2}^{A*D}$. Then, according to the Algorithm 1, where in the role of A is $A*D$, in the role of B is $B*D$, in the role of B_1 is D_2, in the role of P_1 is D_4, have

$$\left\{ \begin{array}{l} \mathbf{1}.\ D_2 \notin OI, \\ \mathbf{2}.\ \ell_{OI}^{D_2} \cap \ell_{OD_2}^{A*D} = D_4, \\ \mathbf{3}.\ \ell_{D_2(B*D)}^{D_4} \cap OI = C. \end{array} \right\} \Longrightarrow A*B + B*D = \ell_{D_2(B*D)}^{D_4} \cap OI$$
$$\Longrightarrow \left\{ \begin{array}{c} OD_2 \parallel (A*D)\, D_4 \\ (B*D)\, D_2 \parallel [(A*B)+(B*D)]\, D_4 \end{array} \right\} \tag{4.12}$$

From (4.9), (4.10) and (4.11) we have

$$IB_1 \parallel AD_1 \parallel BD_2 \parallel (A+B)\, D_5 \parallel (A+B)\, D_3 \tag{4.13}$$

which indicates that the points $A+B, D_3, D_5$, are collinear points. Consequently $D_3 \in (A+B)\, D_5$, that brings

$$AD_1 \parallel D_3 D_5 \tag{4.14}$$

Also, from the (4.9), (4.11) and (4.12), it turns out that

$$\begin{array}{c} D_1 B_1 \parallel (A*D)\, D_1 \parallel (B*D)\, D_2 \parallel [(A+B)*D]\, D_5 \parallel \\ \parallel [(A*D)+(B*D)]\, D_4 \end{array} \tag{4.15}$$

We note that three-vertices $A(A*D)D_1$ and $D_3D_4D_5$ meet the conditions of the Desargues axiom (D1) (Proposition 1.3), since, from the (4.10) and (4.12), we have that:

$$D_1 D_5 \parallel AD_3 \parallel (A*D)\, D_4. \tag{4.16}$$

Therefore from (4.10) and (4.14) we have

$$\begin{matrix} A(A*D) \parallel D_3D_4 \\ AD_1 \parallel D_3D_5 \end{matrix} \overset{\mathbf{D1}}{\Longrightarrow} (A*D)D_1 \parallel D_4D_5 \tag{4.17}$$

Whereas, from (4.15) and (4.17) the resulting that also points $(A+B)*D, D_4, D_5$ are collinear points. Consequently $D_4 \in [(A+B)*D]\, D_5 \parallel DB_1$, that implies

$$[(A+B)*D]\, D_5 \parallel [(A*D)+(B*D)]\, D_4.$$

Namely

$$(A+B)*D = (A*D)+(B*D)$$

(ii) The proof of equation (ii) is analog. However, we present another proof, accepting that, is in similar way with the proof in point a) of equation $(I+B)*D = D+B*D$, it is also a proof of the equation

$$A*(I+D) = A + A*D \tag{4.18}$$

in the case where $A, B, D \neq O$ and $A \neq B \neq D$. In this case, since $B \neq O$, from (3.8), exists the point B^{-1}. Then:

$$\begin{aligned} [A*(B+D)]*B^{-1} &\overset{(6)}{=} A*\left[(B+D)*B^{-1}\right] \\ &\overset{(4.1.i)}{=} A*(B*B^{-1}+D*B^{-1}) \\ &\overset{(3.8)}{=} A*(I+D*B^{-1}) \\ &\overset{(4.18)}{=} A+A*(D*B^{-1}) \\ &\overset{(3.1),(3.3)}{=} A*I+(A*D)*B^{-1} \\ &\overset{(3.8)}{=} A*(B*B^{-1})+(A*D)*B^{-1} \\ &\overset{(3.2)}{=} (A*B)*B^{-1}+(A*D)*B^{-1} \\ &\overset{(4.1.i)}{=} [(A*B)+(A*D)]*B^{-1} \end{aligned}$$

From here we have

$$\begin{aligned} [A*(B+D)]*B^{-1} &= [(A*B)+(A*D)]*B^{-1} \Longrightarrow \\ A*(B+D) &= (A*B)+(A*D)\,. \end{aligned}$$

□

Bearing in mind the Theorem 2 and the Propositions 3.1, 3.2 and 4.1 we obtain this

Theorem 5. *In Desargues affine plane the algebra* $(OI, +, *)$ *is the unitary ring.*

Theorem 6. *In Desargues affine plane the algebra* $(OI, +, *)$ *is a corp (skew field).*

Proof. Since $I \neq O$, in the ring OI has at least one non-zero element. Then, by definition of the a skew-fields (see [5, 19, 20, 21]) requested to prove as follows:

1. $OI^* = OI - \{O\}$, is stable subset of OI about multiplication. To really, if the points $A, B \in OI^*$, then also $A * B \in OI^*$. We suppose $A * B = O$. For as much as $A \neq O$, by (3.2) and (3.8) we have $B = I * B = (A^{-1} * A) * B = A^{-1} * (A * B) = A^{-1} * O = O$.

This contradicts the condition that $B \neq O$.

2. The groupoid $(OI^*, *)$ is a group, because it is a subgroup of $(OI, *)$ which, according to Theorem 4, is a group. □

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