Abstract. In the paper Fractional Multistep differential transform method it is used for investigation of the the approximate numerical solutions of the fractional-order Rössler chaotic system.

Numerical results are presented graphically, with bifurcation diagrams and Lyapunov exponents, for different values of the parameters.

1. Introduction

The basic mathematical ideas about fractional calculus were developed long ago by the Leibniz (1695), Liouville (1834), Riemann (1892), and others. First book on the topic was published by Oldham and Spanier in 1974 ([13]). Recently, fractional calculus has application in physics, signal processing, electromagnetics, bioengineering, etc.

In [8], [7] Multistep differential transform method (MDTM) is applied to fractional-order Rössler chaotic and hyperchaotic systems, and in [9] to fractional-order Chua’s system.

More generaly about DTM one can see in [2].

Twodimensional generalized DTM is used in [12], [10], [11], [14].

2. Fractional Differential Transform Method (FDTM)

Caputo fractional derivative of order \( q, q > 0 \), of the function \( f(x) \) is defined as

\[
D_x^q f(x) = \begin{cases} 
\frac{1}{\Gamma(m-q)} \int_{x_0}^{x} \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, & m - 1 < q < m \\
\frac{d^m}{dx^m} f(x), & q = m 
\end{cases}
\]

where \( x > x_0 \), \( m \) is pozitive integer, \( m - 1 < q \leq m \) and \( \Gamma \) is the Gamma function.

2000 Mathematics Subject Classification. 34C28, 34A08, 74H15.

Key words and phrases. Fractional Rössler system, Multistep differential transform method, Lyapunov exponents.
More details about Caputo fractional derivative one can see, for example, in [3], [15], [4].

Further we assume that \( q > 0 \) is rational number and \( m = 1 \).

The fractional differential transform method consists in the following.

The function \( f(x) \) is expanded in terms of a fractional power series

\[
f(x) = \sum_{k=0}^{\infty} F(k)(x-x_0)^{k/\alpha}
\]

where \( F(k) \) is the fractional differential transform of \( f(x) \) and \( \alpha \) is positive integer such that \( q\alpha \) is positive integer. Usually \( \alpha \) is chosen as smallest number with that property.

Let consider the equation

\[
D_{x_0}^q f(x) = G(x, f)
\]

with initial conditions \( f(x_0), f'(x_0), \ldots, f^{(m-1)}(x_0) \).

In Caputo derivative the initial conditions are implemented to the integer order derivatives, so the transformation of the initial conditions are

\[
F(k) = \begin{cases} 
\frac{1}{(\frac{k}{\alpha})!} \left. \frac{d^{\frac{k}{\alpha}} f(x)}{dx^{\frac{k}{\alpha}}} \right|_{x=x_0}, & k \in \mathbb{Z}^+, k = 0, 1, \ldots, q\alpha - 1 \\
0, & k \alpha \notin \mathbb{Z}^+
\end{cases}
\]

where \( \mathbb{Z}^+ \) is the set of nonnegative integers.

For properties and details about DTM one can see, for example, [1], [6]. Theorems 1-6 from [1] give the necessary properties for computing the coefficients \( F(k) \), for \( k \geq q\alpha \).

3. The Multistep Fractional Differential Transform Method (MFDTM)

The FDTM is not suitable for large intervals ([8]), so we divide the working interval \( [x_0, T] \) into subintervals \( [t_0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N] \), where \( t_0 < t_1 < \ldots < t_N, t_0 = x_0, t_N = T \) and \( t_i - t_{i-1} = h > 0 \), for \( i = 1, \ldots, N \). Then FDTM is applied on first interval with initial condition \( c_1 = f(x_0) \) and it is obtained the approximate solution \( f_1(x) \). At the second interval again FDTM is applied with initial condition \( c_2 = f_1(t_1) \) and it is obtained solution \( f_2(x) \). This procedure is applied to all intervals \( [t_i, t_{i-1}], i = 1, \ldots, N \).

So the final approximate solution is

\[
f(x) = \begin{cases} 
f_1(x), & x \in [t_0, t_1] \\
f_2(x), & x \in [t_1, t_2] \\
\vdots & \\
f_N(x), & x \in [t_{N-1}, t_N].
\end{cases}
\]
Further, we assume that $x_0 = 0$.

4. **Bifurcation analysis of fractional-order Rössler chaotic system**

We consider the fractional-order Rössler chaotic system

\[
\begin{align*}
D^{q_1}x &= -y - z \\
D^{q_2}y &= x + ay \\
D^{q_3}z &= b + z(x - c)
\end{align*}
\]

with initial conditions $x_0 = x(0)$, $y_0 = y(0)$ and $z_0 = z(0)$, where $x, y, z$ are unknown functions of $t$, $q_1 = q_2 = q_3 = q$, $0 < q \leq 1$ and $a, b, c > 0$.

Working interval $[0, T]$, it is divided into subintervals $[0, t_1], [t_1, t_2], \ldots, [t_{N-1}, t_N]$, where $0 < t_1 < t_2 < \ldots t_N = T$ and $t_i - t_{i-1} = h > 0$, for $i = 1, \ldots, N$.

When DTM is applied to this system at the interval $[0, t_1]$ are obtained approximations $x_1, y_1$ and $z_1$ of the functions $x(t), y(t)$ and $z(t)$, respectively.

We apply DTM at the interval $[t_{i-1}, t_i]$, with initial conditions $x_{i-1}(t_{i-1}), y_{i-1}(t_{i-1})$ and $z_{i-1}(t_{i-1})$, and obtain the approximations $x_i, y_i$ and $z_i$:

\[
\begin{align*}
x_i(t) &= \sum_{k=0}^{M_i} X_i(k) (t - t_i)^{k\alpha} \\
y_i(t) &= \sum_{k=0}^{M_i} Y_i(k) (t - t_i)^{k\alpha} \\
z_i(t) &= \sum_{k=0}^{M_i} Z_i(k) (t - t_i)^{k\alpha}
\end{align*}
\]

for every $i = 1, \ldots, N$.

Finally, in this way we obtain approximations $x_N, y_N$ and $z_N$ at the interval $[t_{n-1}, t_N]$.

So, the approximations of functions $x, y$ and $z$ at $[0, T]$ are

\[
\begin{align*}
X(t) &= \begin{cases}
x_1(t), & t \in [0, t_1] \\
x_2(t), & t \in [t_1, t_2] \\
\vdots \\
x_N(t), & t \in [t_{N-1}, t_N]
\end{cases} \\
Y(t) &= \begin{cases}
y_1(t), & t \in [0, t_1] \\
y_2(t), & t \in [t_1, t_2] \\
\vdots \\
y_N(t), & t \in [t_{N-1}, t_N]
\end{cases}
\]

\[
Z(t) = \begin{cases}
z_1(t), & t \in [0, t_1] \\
z_2(t), & t \in [t_1, t_2] \\
\vdots \\
z_N(t), & t \in [t_{N-1}, t_N]
\end{cases}
\]
and

\[
Z(t) = \begin{cases} 
  z_1(t), & t \in [0,t_1] \\
  z_2(t), & t \in [t_1,t_2] \\
  \vdots \\
  z_N(t), & t \in [t_{N-1},t_N] 
\end{cases}
\]

Through this paper the initial conditions are choosen \( x_0 = 2, y_0 = 3 \) and \( z_0 = 4 \), and \( M_1 = M_2 = \cdots = M_N \).

In this section we choose \( \alpha = 10 \), so tranformation of the initial conditions are \( X_i(k) = 0, Y_i(k) = 0 \) and \( Z_i(k) = 0 \), for each interval \( [t_{i-1},t_i] \), \( i = 1,\ldots,N \), and for \( k = 1,\ldots,10q - 1 \).

For \( k = 0 \) initial conditions are \( X_1(0) = 2, Y_1(0) = 3, Z_1(0) = 4 \), and \( X_i(0) = X_{i-1}(t_i), Y_i(0) = Y_{i-1}(t_i), Z_i(0) = Z_{i-1}(t_i) \) for \( i = 1,\ldots,N \).

Using Theorems 1-6 from [1] we obtain that fractional-order Rössler chaotic system, at the interval \( [t_i,t_{i-1}] \) transforms to

\[
\begin{align*}
X_i(k+10q) &= \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(q+1+\frac{1}{\alpha})} (-Y_i(k) - Z_i(k)) \\
Y_i(k+10q) &= \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(q+1+\frac{1}{\alpha})} (X_i(k) + aY_i(k)) \\
Z_i(k+10q) &= \frac{\Gamma(1+\frac{q}{\alpha})}{\Gamma(q+1+\frac{1}{\alpha})} \left( b\delta(k) - cZ_i(k) + \sum_{p=0}^{k} Z_i(p) X_i(k-p) \right)
\end{align*}
\]

where \( \Gamma \) is the Gamma function.

Trough the paper we choose \( h = \frac{1}{100}, T = 400 \) (so \( N = 40000, 0 \leq t \leq 400 \)), \( a = 0.2 \) and \( c = 5.7 \). The parameter \( b \) is changing with stepsize of 0.005 and \( 0 < b \leq 4 \).

All computations are made for \( 0 \leq t \leq 400 \), and the diagrams are ploted for \( 200 \leq t \leq 400 \), using software Mathematica. With maxx (maxy, maxz) are denoted graphs of maximal peaks of \( \{ X(k) | 20001 \leq k \leq 39999 \} \) \( \{ Y(k) | 20001 \leq k \leq 39999 \} \), \( \{ Z(k) | 20001 \leq k \leq 39999 \} \), where

\[
\begin{align*}
  X(k) &= x_k(t_k), & \quad Y(k) &= y_k(t_k), & \quad Z(k) &= z_k(t_k)
\end{align*}
\]

for \( k = 1,\ldots,N \).

Respectively are denoted minimal peaks with minx (miny, minz).

Graphs for maximal Lyapunov exponents are denoted by maxlyapunov.

For \( q = 0.1, q = 0.2 \) and \( q = 0.3 \) occurs overflow in the computations.

- \( q = 0.4 \)

Overflow in computation is occured for \( b < 0.93 \). It is choosen \( M = 10q\alpha \). Rezulting diagrams are showed in Figure 1.
Figure 1: Diagram for $q = 0.4$, $a = 0.2$, $c = 5.7$ and $0 < b < 4$ with stepsize 0.005. Overflow in computation is occurred for $b < 0.93$

- $q = 0.5$
- $M = 5q\alpha$ (Figure 2)
Figure 2: Diagram for $q = 0.5$, $a = 0.2$, $c = 5.7$ and $0 < b < 4$ with stepsize $0.005$.

- $q = 0.7$

Here $M = 5qa$. Resulting diagrams are showed in Figure 3.
Figure 3: Diagram for $q = 0.7$, $a = 0.2$, $c = 5.7$ and $0 < b < 4$ with stepsize 0.005.

- $q = 0.9$
- $M = 5q\alpha$ (Figure 4)
Figure 4: Diagram for $q = 0.9, a = 0.2, c = 5.7$ and $0 < b < 4$ with stepsize 0.005.

Now we change $q$ from $q = 0.01$ to $q = 0.99$ with stepsize $h = 0.01$, and for $a = 0.2, b = 0.2$ and $c = 5.7$. Notations maxx, maxy, minx and miny remains same as before.

In the Figure 5 it is shown dependence maxx and minx from $q$, and in the figure 6 dependence maxy and miny from $q$. 
Figure 5: Diagram for \( \max x \) and \( \min x \), \( a = 0.2, b = 0.2, c = 5.7 \) and \( 0.01 \leq q \leq 0.99 \) with stepsize 0.01.

Figure 6: Diagram for \( \max y \) and \( \min y \), \( a = 0.2, b = 0.2, c = 5.7 \) and \( 0.01 \leq q \leq 0.99 \) with stepsize 0.01.
REFERENCES


Institute of Mathematics, Faculty of Natural Sciences and Mathematics, Sts. Cyril and Methodius University, Skopje, Macedonia

E-mail address: gorgim@pmf.ukim.mk