A STATEMENT OF DIFFERENTIAL CALCULUS IN CONTEXT OF CODOMAIN OF THE FUNCTION

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Abstract. In this article we give an analogue of Statement 3 from [1] with appropriate geometrical interpretation in sense of codomain (image) of the function $f(x)$.


In this article using Lemma 2 [2] of D. Trahan and Corollary 3 from [3], the following Statement, with the mentioned emphasis on the geometrical interpretation, is given.

Statement. Let $f : [a, b] \longrightarrow R$ be differentiable function. If there exists a point $x_0 \in (a, b)$, such that $f''(x_0)$ exists and

\begin{equation}
\left( f'(r) - \frac{f(r) - f(x_0)}{r - x_0} \right) \left( \frac{f(b) - f(a)}{b - a} - \frac{f(r) - f(x_0)}{r - x_0} \right) > 0, \text{ for } r = a \text{ and } r = b,
\end{equation}

then exist at least two values $\xi \in (a, b)$ such that

\begin{equation}
f'(\xi_i) = \frac{f(\xi_i) - f(x_0)}{\xi_i - x_0}, \quad i = 1, 2.
\end{equation}

Proof. Let us introduce function

\begin{equation}
F(x) = \begin{cases} 
\frac{f(x) - f(x_0)}{x - x_0} & : x \neq x_0 \in (a, b) \\
f'(x_0) & : x = x_0
\end{cases}
\end{equation}

It is true, for $x \neq x_0$:

\[ F'(x) = \frac{f'(x)(x - x_0) - (f(x) - f(x_0))}{(x - x_0)^2} \]
and
\[ \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} - \frac{f'(x_0)}{2(x - x_0)} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{2(x - x_0)} = \lim_{x \to x_0} \frac{f''(x)}{2} = \frac{f''(x_0)}{2}, \]

ie. function \( F(x) \) is differentiable over segment \([a, b]\) and \( F'(x_0) = \frac{f''(x_0)}{2} \).

10. Based on the sign of the difference \( \frac{f(b) - f(a)}{b - a} - \frac{f(r) - f(x_0)}{r - x_0} > 0 \), in \( r = a \), we obtained one upper (lower) bound for value \( F(a) \) as follows

\[ (f(b) - f(a))(a - b + b - x_0) < (f(a) - f(b) + f(b) - f(x_0))(b - a), \]

ie.
\[ (f(b) - f(a))(b - x_0) < (f(b) - f(x_0))(b - a). \]

meaning that
\[ \frac{f(b) - f(a)}{b - a} < \frac{f(b) - f(x_0)}{b - x_0} = F(b); \]

and finally

\[ F(a) = \frac{f(a) - f(x_0)}{a - x_0} < \frac{f(b) - f(a)}{b - a} < \frac{f(b) - f(x_0)}{b - x_0} = F(b). \]

Therefore

\[ F(a) < F(b) \iff (F(b) - F(a)) > 0. \]

20. Based on the sign of the difference \( \frac{f(b) - f(a)}{b - a} - \frac{f(r) - f(x_0)}{r - x_0} < 0 \), in \( r = b \), we obtained one lower (upper) bound for value \( F(b) \) as follows

\[ \frac{f(b) - f(a)}{b - a} > \frac{f(b) - f(x_0)}{b - x_0} = F(b), \]

Previous inequality is equivalent with
\[ (f(b) - f(a))(b - a + a - x_0) > (f(b) - f(a) + f(a) - f(x_0))(b - a), \]
ie.

\[(f(b) - f(a))(x_0 - a) \lessgtr (f(x_0) - f(a))(b - a).\]

meaning that

\[F(a) = \frac{f(b) - f(a)}{b - a} \lessgtr \frac{f(b) - f(x_0)}{b - x_0} = F(b);\]

and finally

\[F(a) = \frac{f(a) - f(x_0)}{a - x_0} \gtrless \frac{f(b) - f(a)}{b - a} \gtrless \frac{f(b) - f(x_0)}{b - x_0} = F(b).\]

Therefore

\[(8) \quad F(a) \gtrless F(b) \iff (F(b) - F(a)) \lessgtr 0.\]
Based on the sign of the difference $f'(r) - \frac{f(r) - f(x_0)}{r - x_0} > 0$, in $r = a$, we obtained

$$F'(a) = \frac{f'(x)(x - x_0) - (f(x) - f(x_0))}{(x - x_0)^2} \Big|_{x=a}$$

$$= \frac{f'(a)(a - x_0) - (f(a) - f(x_0))}{(a - x_0)^2}$$

$$= \frac{1}{a - x_0} \left( f'(a) - \frac{f(a) - f(x_0)}{a - x_0} \right) < 0,$$

ie.

(9) \hspace{1cm} F'(a) < 0.

Analogously, in $r = b$, we obtained

$$F'(b) = \frac{1}{b - x_0} \left( f'(b) - \frac{f(b) - f(x_0)}{b - x_0} \right) > 0,$$

ie.

(10) \hspace{1cm} F'(b) > 0.

From (6) and (9) follows

(11) \hspace{1cm} F'(a)(F(b) - F(a)) < 0,

and analogously, based on (8) and (10), we can conclude

(12) \hspace{1cm} F'(b)(F(b) - F(a)) < 0.

Therefore, is true

(13) \hspace{1cm} F'(a)(F(b) - F(a)) < 0 \land F'(b)(F(b) - F(a)) < 0.

Next, using Lemma 2 [2] and Corollary 3 [3] respectively, existence least two $\xi \in (a, b)$ follows such that $F'(%(\xi) = 0$ and therefore follows (in Fletts denotation):

(14) \hspace{1cm} (\exists \xi_i \in (a, b)) f'(\xi_i) = \frac{f(\xi_i) - f(x_0)}{\xi_i - x_0}, \hspace{1cm} i = 1, 2.$$

Geometrically interpretation. In this part we geometrically determined part of codomain of the function $f$ over segment $[a, b]$ in case of tangents in points $A$ and $B$ and their intersection ($f'(a) \neq f'(b)$), based on the following conditions:

(15) \hspace{1cm} f'(a) - \frac{f(x_0) - f(a)}{x_0 - a} > 0 \land f'(b) - \frac{f(b) - f(x_0)}{b - x_0} > 0.$
From previous proof of the Statement we have:
\[
\frac{f(x_0) - f(a)}{x_0 - a} < f'(a) \quad \land \quad f'(b) > \frac{f(b) - f(x_0)}{b - x_0},
\]
ie, \( (16) \)
\( f(x_0) < T_A: f(x_0) < f(a) + f'(a)(x_0 - a) \land f(x_0) > T_B: f(x_0) > f(b) + f'(b)(x_0 - b); \)
therefore we have the following Figure of function \( f(x) \) over segment \([a, b]\) and tangent in point \( A \) (bellow this tangent is region \( D_1 \)), and tangent in point \( B \) (above this tangent is region \( D_2 \)). On this way in Figure we have interior region \( D = D_1 \cap D_2 \) for both possibility in \((14)\). Special consideration in connection of codomain of the function \( f(x) \) in regions \( D_1 \) and \( D_2 \) correspond relations \((11)\) and \((12)\) respectively.

\[
\text{Figure}
\]

REFERENCES

ЕДНО ТВРДЕЊЕ ЗА ДИФЕРЕНЦИЈАЛНОТО СМЕТАЊЕ ВО КОНТЕКСТ НА КОДОМЕНОТ НА ФУНКЦИЈАТА

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Р е з и м е

 Во овој труд даваме аналогно тврдење на тврдењето 3 од [1] со соодветна геометриска интерпретација во смисла на кодоменот (сликата) на функцијата $f(x)$.

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