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COMPACTNESS OF $\mathcal{S}(n)$-CLOSED SPACES

IVAN LONČAR

Abstract. The aim of this paper is to study compactness of the $\mathcal{S}(n)$-closed spaces. It is proved that $S(n)$-closed space $(X,\ \tau)$ is compact if every closed subset of $(X,\ \tau)$ is $S(n)$-set and that sequentially $S(n)$-closed space X is countably compact if every closed subset of X is θ^n-*closed.*

1. Introduction

Let $\mathcal{P}$ be a class of topological spaces. A space $X \in \mathcal{P}$ is said to be $\mathcal{P}$-*closed* iff X is closed in every $\mathcal{P}$ space in which it is embedded.

In this paper we shall study compactness of $\mathcal{S}(n)$-closed spaces. The symbol $\mathbb{N}^+$ denotes the set of positive integers and $\mathbb{N} = (0) \cup \mathbb{N}^+$.

Introduction contains the well-known characterizations of compact spaces which we need in the remaining sections.

Let R be a family of sets that together with A and B contains the intersection $A \cap B$. By a *filter in* R [3, pp. 124-133] we mean a non-empty subfamily $\mathcal{F} \subset R$ satisfying the following conditions:

(Fl) $\emptyset \notin \mathcal{F}$.

(F2) *If* $A_1,\ A_2 \in F$, *then* $A_1 \cap A_2 \in \mathcal{F}$.

(F3) *If* $A \in \mathcal{F}$ *and* $A \subset A_1 \in R$, then $A_1 \in \mathcal{F}$.

A filter $\mathcal{F}$ in R is a *maximal filter* or an *ultrafilter in* R, if for every filter $\mathcal{F}'$ in R that contains $\mathcal{F}$ we have $\mathcal{F}' = \mathcal{F}$.

A *filter-base* in R is a non-empty family $g \subset R$ such that $\emptyset \notin g$ and

(FB) *If* $A_1,\ A_2 \in g$, *then there exists an* $A_3 \in g$ such *that* $A_3 \subset A_1 \cap A_2$.

One readily sees that for any filter-base $\mathcal{F}$ in R, the family

$$\mathcal{F}_g = \{\ A \in R\text{: there exists a } B \in g \text{ such that } B \subset A\}$$

is a filter in R.

By a *filter* (a *filter-base*) *in a topological* space X we mean a filter (a filter-base) in the family of all subsets of X.

A point x is called a *limit of a filter* $\mathcal{F}$ if every neighbourhood of x belongs to $\mathcal{F}$; we then say that the filter F *converges* to x and we write $x \in \lim \mathcal{F}$. A point x is called a limit *of a filter-base* if $x \in \lim \mathcal{F}_g$; we then

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say that the filter-base g *converges* to x and we write $x \in \lim g$. Clearly, $x \in \lim g$ if and only if every neighbourhood of x contains a member of g.

A point x is called a *cluster point of a filter* $\mathcal{F}$ (*of a filter-basc* g) if x belongs to the closure of every member of $\mathcal{F}$ (of g). Clearly, x is a cluster point of a filter $\mathcal{F}$ (of a filter-base g) if and only if every neighbourhood of x intersects all members of $\mathcal{F}$ (of g). This implies in particular that every cluster point of an ultrafilter is a limit of this ultrafilter.

A *cover* of a set X is a family $\{A_s : s \in S\}$ of subsets of X such that $X = \cup\{A_s : s \in S\}$. $\mathrm{Cov}(X)$ is the set of all coverings of topological space X. We say that a cover $\mathcal{B}$ of space X is *refinement* of a cover $\mathcal{A}$ of the same space if for every $B \in \mathcal{B}$ there exists $A \in \mathcal{A}$ such that $B \subset A$. If $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(X)$ and $\mathcal{V}$ refines $\mathcal{U}$, we write $\mathcal{V} \prec \mathcal{U}$.

Definition. [12, 17.3 Definition, p. 118]A family $\mathcal{F}$ of subsets of X has the *finite intersection property* iff the intersection of any finite subcollection from $\mathcal{F}$ is nonempty.

Remark 1. *Families with finite intersection property are somewhat like filters; in fact, if $\mathcal{G}$ is such a family and $\mathcal{F}$ is the collection of all possible finite intersections from $\mathcal{G}$ then $\mathcal{F}$ is a filter base, so every family $\mathcal{G}$ with finite intersection property generates a filter. Conversely, every filter is a family with finite intersection property.*

Definition 1. *A topological space X is called a quasi-compact space if every open cover of X has a finite subcover, i.e., if for every open cover $\{U_s : s \in S\}$ of the space X there exists a finite set $\{s_1, s_2, ..., s_k\} \subset S$ such that $X = U_{s_1} \cup U_{s_2} \cup ... \cup U_{s_k}$. A space X is a compact space if it is quasi-compact and Hausdorff.*

Lemma 1. *A Hausdorff space X is compact if and only if every open cover of X has a finite refinement.*

Theorem 1. [3, 1.1. THEOREM, p. 124] *A Hausdorff space X is compact if and only if every family of closed subsets of X which has the finite intersection property has non-empty intersection.*

Theorem 2. [12, 17.4 Theorem, p. 118] *For a Hausdorff topological space X, the following are equivalent:*

(1) *X is compact,*
(2) *each family of closed subsets of X with the finite intersection property has nonempty intersection,*
(3) *each filter in X has a cluster point,*
(4) *each net in X has a cluster point,*
(5) *each ultranet in X converges,*
(6) *each ultrafilter in X converges.*

A topological space X is called a *countably compact* space if X is a Hausdorff space and every countable open cover of X has a finite subcover. Thus, every compact space is countably compact; more precisely:

Theorem 3. [3, Theorem 3.10.2, p. 202] *For every Hausdorff space X the following conditions are equivalent:*

- **(i):** *The space X is countably compact.*
- **(ii):** *Every countable family of closed subsets of X which has the finite intersection property has non-empty intersection.*

2. S(n)-closed spaces

The concept of *θ-closure* was introduced by Veličko [10]. For a subset M of a topological space X the *θ-closure* is defined by $\operatorname{Cl}_\theta M = \{x \in X\colon$ every closed neighborhood of x meets $M\}$, M is *θ-closed* if $\operatorname{Cl}_\theta M = M$. This concept was used by many authors for the study of Hausdorff non-regular spaces. The θ-closure is related especially to Urysohn spaces (every pair of distinct points can be separated by disjoint closed neighborhoods). A space X is Urysohn iff the diagonal in $X \times X$ is θ-closed.

We say that a pair (G, H) is an *ordered pair* of open sets about $x \in X$ if G and H are open subsets of X and $x \in G \subset \operatorname{Cl} G \subset H$. A point $x \in X$ is in *u-closure* of a subset $K \subset X$ $(x \in \operatorname{Cl}_u K)$ if each ordered pair (G, H) of open sets about $x \in X$ satisfies $K \cap \operatorname{Cl} H \neq \emptyset$. A subset K of a space X is *u-closed* if $K = \operatorname{Cl}_u K$.

A generalization of the concepts of θ-closure and of u-closure *is θ^n-closure.*

For a positive integer n and a subset M of a topological space X, the *θ^n-closure* $\operatorname{Cl}_{\theta^n} M$ of M is defined to be the set [2]

$$\begin{aligned} &\{x \in X : \text{for every chain of open neighborhoods of } x,\\ &\text{if } U_1 \subset U_2 \subset ... \subset U_n \text{ with } \operatorname{Cl}(U_i) \subset U_{i+1},\\ &\text{where } i = 1, 2, ..., n-1, \text{ then one has } \operatorname{Cl}(U_n) \cap M \neq \emptyset\}. \end{aligned}$$

For $n = 1$ this gives the θ-closure. Moreover, for $n = 2$ the above definition gives u-closure (See Introduction).

Definition 2. *A subset M of X is said to be θ^n-closed if $M = \operatorname{Cl}_{\theta^n} M$. Similarly θ^n-interior of M is defined and denoted by* $\operatorname{Int}_{\theta^n} M$, *so* $\operatorname{Int}_{\theta^n} M = X \setminus \operatorname{Cl}_{\theta^n}(X \backslash M)$.

Proposition 1. *Every θ^n- closed subset $M \subset X$ is closed.*

Proof. See [9, p. 222]. □

Definition 3. *An open set U is called a n-hull of a set A (see* [6, p. 624]*) if there exists a family of open sets $U_1, U_2, ..., U_n = U$ such that $A \subset U_1$ and $\operatorname{Cl} U_i \subset U_{i+1}$ for $i = 1, ..., n-1$.*

Definition.For $n \in \mathbb{N}$ and a filter $\mathcal{F}$ on X we denote by $\operatorname{ad}_{\theta^n}\mathcal{F}$ the set of θ^n- adherent points of $\mathcal{F}$, i.e. $\operatorname{ad}_{\theta^n}\mathcal{F} = \cap\{\operatorname{Cl}_{\theta^n} F_\alpha : F_\alpha \in \mathcal{F}\}$. In particular $\operatorname{ad}_{\theta^0}\mathcal{F} = \operatorname{ad}\mathcal{F}$ is the set of adherent points of $\mathcal{F}$.

Definition.Let X be a space and $n \in \mathbb{N}$; a point x of X is $S(n)$-*separated from a subset* M of X if $x \notin \mathrm{Cl}_{\theta^n} M$. In particular x is $S(0)$-separated from M if $x \notin \mathrm{Cl}\, M$.

Definition 4. *Let* $n \in \mathbb{N}$ *and* X *be a space:*

(a) X is an $S(n)$-*space* if every pair of distinct points of X are S(n)-separated;

(b) a filter $\mathcal{F}$ on X is an $S(n\ )$-*filter* if every nonadherent point of $\mathcal{F}$ is $S(n)-$ separated from some member of $\mathcal{F}$;

(c) an open cover $\{U_\alpha\}$ of X is an $S(n)$-*cover* if every point of X is in the θ^n-interior of some U_α.

The $S(n)$-spaces coincide with the $\overline{T}_n$-spaces defined in [11] and studied further in [7], where also $S(\alpha)$-spaces are defined for each ordinal α.

Proposition 2. *The* $S(0)$*-spaces are the* T_0 *spaces, the* $S(1)$*-spaces are the Hausdorff spaces and the* $S(2)$*-spaces are the Urysohn spaces.*

Clearly every *filter* is an $S(0)-$ *filter,* every open cover is an $S(0)$-cover and every open filter is an $S(1)$-filter. The open $S(2)$-filters coincide with the Urysohn filters defined in [5] and [8]. For $n \geq 1$ the open $S(n)$-filters were defined in [7]. The special covers used in (3.9) [7] are $S(n-1)$ covers, $S(2)$-covers are the Urysohn covers defined in [1]. In a regular space every filter (resp. open cover) is an S(n)-filter (resp. $S(n)$-cover) for every $n \in \mathbb{N}$.

The following Proposition plays fundamental role.

Proposition 3. *In any topological space:*

- **a):** *the empty set and the whole space are* Θ^n*-closed,*
- **b):** *arbitrary finite unions of* Θ^n*-closed sets are* Θ^n*-closed,*
- **c):** *arbitrary intersection of* Θ^n*-closed sets are* Θ^n*-closed,*
- **d):** *a* Θ^n*-closed subset is closed,*
- **e):** $\mathrm{Cl}\, K \subset \mathrm{Cl}_{\Theta^n} K$ *for each subset* K.

Proof. **a)** By definition.

b) Let $F = \cup\{F_i : i = 1, ..., n\}$ where each F_i is Θ^n-closed. For each $x \notin F$ there exist n-hull U_i of x such that $\mathrm{Cl}\, U_i \cap F_i = \emptyset$, $i = 1..., n$. Now $U = \cap\{U_i : i = 1, ..., n\}$ is n-hull of x such that $\mathrm{Cl}\, U \cap F = \emptyset$. This means that $x \notin \mathrm{Cl}_{\theta^n} F$, i.e. F is Θ^n-closed.

c) Assume that $x \in \mathrm{Cl}_{\theta^n} F$, where $F = \cap\{F_\alpha : \alpha \in A\}$ and each F_α is θ^n- closed. This means that for each n-hull U of the point x we have $\mathrm{Cl}\, U \cap F \neq \emptyset$. Clearly $\mathrm{Cl}\, U \cap F_\alpha \neq \emptyset$ for every $\alpha \in A$. We infer that $x \in F_\alpha$, $\alpha \in A$, since each F_α is θ^n- closed. Finally, $x \in \cap\{F_\alpha : \alpha \in A\} = F$ and F is θ^n- closed ($F = \mathrm{Cl}_{\theta^n} F$).

d) See Proposition 1.

e) The set $\mathrm{Cl}\, K$ is minimal closed set containing K. Hence, $\mathrm{Cl}\, K \subset \mathrm{Cl}_{\Theta^n} K$. □

Definition 5. *For a space $(X,\ \tau)$ and $n \in \mathbb{N}$ denote by $(X,\ \tau_{\theta^n})$, where τ_{θ^n} is the topology on X generated by the θ^n-closure, i.e. having as closed sets all θ^n-closed sets in $(X,\ \tau)$.*

Clearly $\tau_{\theta^0} = \tau,\ \tau_{\theta^1} = \tau_\theta$ and a subset U of X is τ_{θ^n}-*open* iff every element of U is contained in the θ^n-interior of U.

The next proposition follows directly from the definitions.

Proposition 4. *For a topological space $(X,\ \tau)$ and $n \in \mathbb{N}^+$ the following conditions are equivalent:*

(a) $(X,\ \tau)$ *is an* $S(n)$*-space,*
(b) $(X,\ \tau_{\theta^n})$ *is a* T_1 *space*
(c) $(X,\ \tau_{\theta^n})$ *is* T_0 and $(X,\ \tau)$ *is* T_1.
If $n \geq 1$, *then these conditions are equivalent to*:
(d) $(X,\ \tau_{\theta^n})$ *is* T_0.
If $n = 2k$ *with* $k \in \mathbb{N}^+$, *then the above conditions are equivalent to*:
(e) *the diagonal in* $X \times X$ *is* θ^k*-closed*

This Proposition show the pivotal role of the topologies τ_{θ^k} in the study of $S(n)$-spaces for $n \geq 1$. In some sense they replace the semiregularization which was the main tool in the study of *H-closed* spaces.Let us observe that we have the following result.

Definition 6. *The identity mapping* $i : (X,\ \tau) \to (X,\ \tau_{\theta^n})$ *is continuous.*

Definition 7. *An S(n)-space* $M, n > 0$, *is S(n)-closed* [2] *if it is* θ^n*- closed in every S(n)-space in which it can be embedded.*

Proposition 5. [2, Proposition 2.1., p. 63] *Let* $n \in \mathbb{N}^+$ *and* X *be a space. Then the following conditions are equivalent:*

(a) *for every open filter* $\mathcal{F}$ *on* X $\ \mathrm{ad}_{\theta^n}\mathcal{F} \neq \emptyset$;
(b) *for every filter* $\mathcal{F}$ *on* X $\ \mathrm{ad}_{\theta^n}\mathcal{F} \neq \emptyset$;
(c) *for every open* $S(n)$*-filter* $\mathcal{F}$ on X $\mathrm{ad}\mathcal{F} \neq \emptyset$;
(d) *for every* $S(n-1)$*-cover* $\{U_\alpha\}$ *of* X *there exist* $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{l_\iota}$ *such that* $X = \bigcup_{i=1}^{k} \mathrm{Cl}\, U_{\alpha_1}$.
If X *is an* $S(n)$*-space then the above conditions are equivalent to*:
(e) X *is* $S(n)$*-closed.*

Lemma 2. *If X is S(n)-closed, then every family* $\{A_\mu,\ \mu \in \Omega\ \}$ *of* θ^n*-closed subsets of X with the finite intersection property has a non-empty intersection* $\cap\{A_\mu,\ \mu \in \Omega\ \}$.

Proof. Let X be S(n)-closed and let $\{A_\mu,\ \mu \in \Omega\ \}$ be a family of θ^n-closed subsets of X with the finite intersection property. Let us recall that $\{A_\mu,\ \mu \in \Omega\ \}$ generates a filter (See Claim 1). By (b) of Proposition 5 we infer that $\ \mathrm{ad}_{\theta^n}\{A_\mu, \mu \in \Omega\} \neq \emptyset$, i.e. $\cap\{\mathrm{Cl}_{\theta^n} A_\mu, \mu \in \Omega\} \neq \emptyset$ 2. But $\mathrm{Cl}_{\theta^n} A_\mu = A_\mu$ since $\{A_\mu,\ \mu \in \Omega\ \}$ is a family of θ^n-closed subsets of X. Finally we infer that $\cap\{A_\mu, \mu \in \Omega\} \neq \emptyset$. □

Lemma 3. *If $(X,\ \tau)$ is $S(n)$-closed space then the space $(X,\ \tau_{\theta^n})$ is quasi-compact.*

Proof. Let $(X,\ \tau)$ be an $S(n)$-closed space and let us prove that $(X,\ \tau_{\theta^n})$ is quasi-compact. For every filter $\mathcal{F} = \{F : F \in \mathcal{F}\}$ of closed sets on $(X,\ \tau_{\theta^n})$ we have the family $\{\mathrm{Cl}_{\Theta_n} i^{-1}(F) : F \in \mathcal{F}\}$ with non-empty intersection $\cap\{\mathrm{Cl}_{\Theta_n} i^{-1}(F) : F \in \mathcal{F}\}$ since $(X,\ \tau)$ is $S(n)$-closed space. It is clear that $\cap\{F : F \in \mathcal{F}\} \neq \emptyset$ since $i(\mathrm{Cl}_{\Theta_n} i^{-1}(F)) = \mathrm{Cl}\, F = F$. Hence, $(X,\ \tau_{\theta^n})$ is quasi-compact. □

Theorem 4. *$S(n)$-closed space $(X,\ \tau)$ is quasi-compact if every closed subset of $(X,\ \tau)$ is θ^n- closed. Moreover, if $n \geq 1$, then $(X,\ \tau)$ is compact.*

Proof. If X is S(n)-closed, then every family $\{A_\mu,\ \mu \in \Omega\ \}$ of θ^n-closed subsets of X with the finite intersection property has a non-empty intersection $\cap\{A_\mu,\ \mu \in \Omega\ \}$ (Lemma 2). Now, let $\{B_\mu,\ \mu \in \Omega_1\}$ be a family closed subsets of X with the finite intersection property. Then $\{\mathrm{cl}_{\theta^n} B_\mu, \mu \in \Omega_1\}$ is a family of θ^n- *closed* sets with the finite intersection property. Hence, $\{\mathrm{cl}_{\theta^n} B_\mu, \mu \in \Omega_1\}$ has a non-empty intersection $\cap\{\mathrm{cl}_{\theta^n} B_\mu,\ \mu \in \Omega\ \}$. It follows that $\cap\{B_\mu,\ \mu \in \Omega\ \} \neq \emptyset$ since $B_\mu = \mathrm{cl}_{\theta^n} B_\mu$. □

Theorem 5. *$S(n)$-closed space $(X,\ \tau)$ is quasi-compact if every closed subset of $(X,\ \tau)$ is $S(n)$-closed. Moreover, if $n \geq 1$, then $(X,\ \tau)$ is compact.*

Proof. If a closed subset F of $(X,\ \tau)$ is $S(n)$-*closed in S(n)-space* $(X,\ \tau)$, then F is θ^n- *closed* (Definition 7). Apply Theorem 4. □

Corollary 1. *H-closed space $(X,\ \tau)$ is compact if every closed subset of $(X,\ \tau)$ is H- closed.*

Proof. A Hausdorff spaces are $S(1)$-spaces 2. Apply Theorem5. □

Corollary 2. *U-closed space $(X,\ \tau)$ is compact if every closed subset of $(X,\ \tau)$ is U- closed.*

Proof. The Urysohn spaces are $S(2)$-spaces. Apply Theorem5. □

Definition 8. *Let X be any topological space, M a subset of X, and let $n \geq 0$.*

- **a):** *A cover $(U_i : i \in I)$ is S(n)-cover with respect to M if $M \subset \cup\{\mathrm{int}_{\Theta^n} U_i : i \in I\}$.*
- **b):** *M is an S(n)-set of X iff every S(n)-cover with respect to M has a finite subcover* [9, Proposition 2.2.].

Proposition 6. *Every H-set of a space X is an $S(n)-$set, for every $n > 0$.*

Proposition 7. *Let M be an $S(n)-$set, $n > 0$, of a space $(X,\ \tau)$. Then $i(M) \subset (X,\ \tau_{\theta^n})$ is compact, where $i : (X,\ \tau) \to (X,\ \tau_{\theta^n})$ is the identity mapping (See 6).*

Proposition 8. *If X is S(n)-closed, then every family $\{A_\mu,\ \mu \in \Omega\ \}$ of S(n)-sets of X $\{A_\mu,\ \mu \in \Omega\ \}$ with the finite intersection property has a non-empty intersection $\cap\{A_\mu,\ \mu \in \Omega\ \}$.*

Proof. By Proposition 3 the space $(X,\ \tau_{\theta^n})$ is quasi-compact. Moreover, each $i(A_\mu) \subset (X,\ \tau_{\theta^n})$ is compact. Now, $\{i(A_\mu),\ \mu \in \Omega\ \}$ is the family with the finite intersection property. It follows that $\cap\{i(A_\mu),\ \mu \in \Omega\ \} \neq \emptyset$. Hence $\cap\{A_\mu,\ \mu \in \Omega\ \} \neq \emptyset$. □

Proposition 9. *$S(n)$-closed space $(X,\ \tau)$ is compact if every closed subset of $(X,\ \tau)$ is $S(n)$-set.*

Proof. Apply Proposition 8. □

3. Sequentially S(n)-closed spaces

A space Y is called *sequentially determined extension* [4] of its subspace X iff for every point $y \in Y$ there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty} x_n = y$. Let $\mathcal{P}$ be a class of topological spaces. A space $X \in \mathcal{P}$ is said to be *sequentially $\mathcal{P}$ -closed* iff X is sequentially closed in every $\mathcal{P}$ space in which it is embedded. In other words X is sequentially $\mathcal{P}$-closed iff X has no sequentially determined extension $Y \in \mathcal{P}$ and $Y \neq X$.

Obviously, every $\mathcal{P}$-closed space is sequentially $\mathcal{P}$ -closed.

Let X be a topological space and $n \in N$. The point $x \in X$ will be called *S^n- limit* (*θ^n- limit*) of a sequence $\{x_n\}_{n=1}^{\infty}$ in X, iff for every chain $U_1 \subset U_2 \subset \ldots \subset U_n$ of *open* neighbourhoods of x such that $\operatorname{cl} U_i \subset U_i + 1$ for $i = 1, 2,\ \ldots,\ n$-l, $U_n(\operatorname{cl} U_n)$ contains all but a finite number of the member of the sequence. Every sequence which has an S^n -limit (θ^n -limit) will be called *S^n-convergent* (*θ^n- convergent*). If for every chain $U_1 \subset U_2 \subset \ldots \subset U_n$ of open neighbourhoods of x such that $U_i \subset U_{i+1}$ for $i = 1, 2,\ \ldots,\ ;\iota - 1,\ U_n(\operatorname{cl} U_n)$ contains infinitely many members of the sequence then x will be called *S^n -adherent point* (*θ^n -adherent point*) of the sequence $\{x_n\}_{n=1}^{\infty}$.

The next theorem characterizes *sequentially S (n) - closed* spaces [4, Theorem 2.1, p. 5].

Theorem 6. *Let X be a T_1 space and $n \in N$. The following conditions are equivalent:*

(a) *every sequence in X has a θ^n-adherent point,*
(b) *every sequence in X has an S^{n+1}-adherent point,*
(c) *every countable $S(n)$-cover of X has a finite subcover,*
(d) *every $S(n)-$filter with a countable base of closed sets has an adherent point,*
(e) *every open elementary $S(n)-$filter has an adherent point,*
(f) *every maximal open elementary $S(n)-$filter has an adherent point,*

If X is an $S(n)$ -space then the above conditions are equivalent to:
(g) *X is sequentially $S(n)$ -closed.*

Lemma 4. *If X is sequentially S(n)-closed, then every countable family $\{A_n,\ n \in \mathbb{N}\ \}$ of θ^n-closed subsets of X with the finite intersection property has a non-empty intersection $\cap\{A_n,\ n \in \mathbb{N}\}$.*

Proof. Suppose that $\cap\{A_n,\ n \in \mathbb{N}\}$ is empty. Then $\{U_n = X \diagdown A_n,\ n \in \mathbb{N}\}$ is countable $S(n)$ -cover of X. By (c) of Theorem 6 we infer that this cover has a finite subcover $\{U_{n_1}, ..., U_{n_k}\}$. From $U_{n_1} \cup ... \cup U_{n_k} = X$ it folows that $X \diagdown U_{n_1} \cap ... \cap X \diagdown U_{n_k} = \emptyset$. We infer that $A_{n_1} \cap ... \cap A_{n_k} = \emptyset$. This contradicts the finite intersection property of $\{A_n,\ n \in \mathbb{N}\ \}$. $\square$

Theorem 7. *Sequentially $S(n)$-closed space X is countably compact if every closed subset of X is θ^n- closed.*

Proof. If X is sequentially S(n)-closed, then every countable family $\{A_n,$ $n \in \mathbb{N}\}$ of θ^n-closed subsets of X with the finite intersection property has a non-empty intersection $\cap\{A_n,\ n \in \mathbb{N}\}$ (Lemma 4). Now, let $\{B_n,\ n \notin \mathbb{N}\}$ be a countable family of closed subsets of X with the finite intersection property. Then $\{\mathrm{cl}_{\theta^n} B_n,\ n \in \mathbb{N}\}$ is a family of θ^n- *closed* sets with the finite intersection property. Hence, $\{\mathrm{cl}_{\theta^n} B_\mu,\ n \in \mathbb{N}\}$ has a non-empty intersection $\cap\{\mathrm{cl}_{\theta^n} B_n,\ n \in \mathbb{N}\ \}$. It follows that $\cap\{B_n,\ n \in \mathbb{N}\ \} \neq \emptyset$ since $B_n = \mathrm{cl}_{\theta^n} B_n$. By (ii) of Theorem 3 we infer that X is countably compact. $\square$

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Faculty of Organizations and Informatics Varaždin, Croatia

E-mail address: ivanloncar18@gmail.com