

**$\mathcal{I}_\theta$  -STATISTICALLY CONVERGENT SEQUENCES IN  
TOPOLOGICAL GROUPS**

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**Abstract.** Recently, Das, Savaş and Ghosal [4], defined the lacunary statistical analogue for the sequence  $x = (x_k)$  as follows: A real sequence  $x = (x_k)$  is said to be  $\mathcal{I}$ -lacunary statistically convergent to  $L$  or  $S_\theta^\mathcal{I}$ -convergent to  $L$  if for any  $\epsilon > 0$  and  $\delta > 0$

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case write  $S_\theta^\mathcal{I} - \lim x = L$  or  $x_k \rightarrow L(S_\theta^\mathcal{I})$ .

In this paper, we introduce and study  $\mathcal{I}$ -lacunary statistical convergence for sequence in topological groups and we shall also present some inclusion theorems.

## 1. INTRODUCTION

We begin this section by giving some preliminaries.

The notion of statistical convergence, which is an extension of the idea of usual convergence, was introduced by Fast [7] and Schoenberg [26] and its topological consequences were studied first by Fridy [8] and Šalát [20]. Di Maio and Kočinac [16] introduced the concept of statistical convergence in topological spaces and statistical Cauchy condition in uniform spaces and established the topological nature of this convergence. The notion has also been defined and studied in different steps, for example, in the locally convex space [15]; in intuitionistic fuzzy normed spaces [18]. In [1] Albayrak and Pehlivan studied this notion in locally solid Riesz spaces. Recently, in [17] Mohiuddine and et.al introduced the concept of lacunary statistical convergence, lacunary statistically bounded and lacunary statistically Cauchy in the framework of locally solid Riesz spaces. Quite recently, Das and Savaş [5] introduced the ideas of  $\mathcal{I}_\tau$ -convergence,  $\mathcal{I}_\tau$ -boundedness and  $\mathcal{I}_\tau$ -Cauchy condition of nets in a locally solid Riesz space.

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The notion of statistical convergence depends on the density of subsets of  $\mathbf{N}$ . A subset  $E$  of  $\mathbf{N}$  is said to have density  $\delta(E)$  if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_E(k) \text{ exists.}$$

Note that if  $K \subset \mathbf{N}$  is a finite set, then  $\delta(K) = 0$ , and for any set  $K \subset \mathbf{N}$ ,  $\delta(K^C) = 1 - \delta(K)$ .

**Definition 1.1.** A sequence  $x = (x_k)$  is said to be *statistically convergent* to  $\ell$  if for every  $\varepsilon > 0$

$$\delta(\{k \in \mathbf{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write  $st - \lim_k x_k = L$  and we denote the set of all statistical convergent sequences by  $st$ .

By a lacunary sequence, we mean an increasing sequence  $\theta = (k_r)$  of positive integers such that  $k_0 = 0$  and  $h_r : k_r - k_{r-1} \rightarrow \infty$  as  $r \rightarrow \infty$ . Throughout this paper, the intervals determined by  $\theta$  will be denoted by  $I_r = (k_{r-1}, k_r)$ , and the ratio  $(k_r)(k_{r-1})^{-1}$  will be abbreviated by  $q_r$ .

In another direction, in [9], a new type of convergence called lacunary statistical convergence was introduced as follows: A sequence  $(x_k)$  of real numbers is said to be lacunary statistically convergent to  $L$  (or,  $S_\theta$ -convergent to  $L$ ) if for any  $\epsilon > 0$ ,

$$\lim_{r \rightarrow \infty} \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| = 0$$

where  $|A|$  denotes the cardinality of  $A \subset \mathbf{N}$ . In [9] the relation between lacunary statistical convergence and statistical convergence was established among other things. In [18], Mursaleen and Mohiuddine extended the idea of lacunary statistical convergence with respect to the intuitionistic fuzzy normed space.

In [6], P. Kostyrko et al. introduced the concept of  $\mathcal{I}$ -convergence of sequences in a metric space and studied some properties of such convergence. Note that  $\mathcal{I}$ -convergence is an interesting generalization of statistical convergence. More investigations in this direction and more applications of ideals can be found in [4, 6, 11, 14, 21, 22, 23, 24, 25].

The purpose of this paper is to study  $\mathcal{I}$ -lacunary statistical convergence of sequences in topological groups and to give some important inclusion theorems.

## 2. DEFINITIONS AND NOTATIONS

The following definitions and notions will be needed.

**Definition 2.1.** A family  $\mathcal{I} \subset 2^{\mathbf{N}}$  is said to be an ideal of  $\mathbf{N}$  if the following conditions hold:

- (a)  $A, B \in \mathcal{I}$  implies  $A \cup B \in \mathcal{I}$ ,
- (b)  $A \in \mathcal{I}$ ,  $B \subset A$  implies  $B \in \mathcal{I}$ .

**Definition 2.2.** A non-empty family  $F \subset 2^{\mathbb{N}}$  is said to be an ideal of  $\mathbb{N}$  if the following conditions hold:

- (a)  $\phi \notin F$ ,
- (b)  $A, B \in F$  implies  $A \cap B \in F$ ,
- (c)  $A \in F$ ,  $A \subset B$  implies  $B \in F$ .

If  $\mathcal{I}$  is a proper ideal of  $\mathbb{N}$  (i.e.,  $\mathbb{N} \notin \mathcal{I}$ ), then the family of sets

$$F(\mathcal{I}) = \{M \subset \mathbb{N} : \exists A \in \mathcal{I} : M = \mathbb{N} \setminus A\}$$

is a filter of  $\mathbb{N}$ . It is called the *filter associated with the ideal*.

**Definition 2.3.** A proper ideal  $\mathcal{I}$  is said to be *admissible* if  $\{n\} \in \mathcal{I}$  for each  $n \in \mathbb{N}$ .

Throughout the paper,  $\mathcal{I}$  will stand for a proper admissible ideal of  $\mathbb{N}$  and by sequence we always mean sequences of real numbers.

**Definition 2.4** (See [6]). Let  $\mathcal{I} \subset 2^{\mathbb{N}}$  be a proper admissible ideal in  $\mathbb{N}$ .

- (i) The sequence  $\{x_k\}$  of elements of  $\mathbb{R}$  is said to be  $\mathcal{I}$ -convergent to  $L \in \mathbb{R}$  if for each  $\epsilon > 0$  the set  $A(\epsilon) = \{n \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}$ .
- (ii) The sequence  $\{x_k\}$  of elements of  $\mathbb{R}$  is said to be  $I^*$ -convergent to  $L \in \mathbb{R}$  if there exists  $M \in F(\mathcal{I})$  such that  $\{x_k\}_{k \in M}$  converges to  $L$ .

In [4], Das, Savaş and Ghosal defined  $\mathcal{I}$ -statistical convergence and  $\mathcal{I}$ -lacunary statistical convergence as follows:

**Definition 2.5.** A sequence  $x = \{x_k\}$  is said to be  $\mathcal{I}$ -statistically convergent to  $L$  or  $S(\mathcal{I})$ -convergent to  $L$  if for each  $\epsilon > 0$  and  $\delta > 0$

$$\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write  $x_k \rightarrow L(S(\mathcal{I}))$ . The class of all  $\mathcal{I}$ -statically convergent sequences will be denoted by simply  $S(\mathcal{I})$ .

**Definition 2.6.** Let  $\theta$  be a lacunary sequence. A sequence  $x = \{x_k\}$  is said to be  $\mathcal{I}$ -lacunary statistically convergent to  $L$  or  $S_\theta^{\mathcal{I}}$ -convergent to  $L$  if for any  $\epsilon > 0$  and  $\delta > 0$

$$\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\} \in \mathcal{I}.$$

In this case we write  $x_k \rightarrow L(S_\theta^{\mathcal{I}})$ . The class of all  $\mathcal{I}$ -lacunary statistically convergent sequences will be denoted by  $S_\theta^{\mathcal{I}}$ .

By  $X$ , we will denote an abelian topological Hausdorff group, written additively, which satisfies the first axiom of countability. For a subset  $A$  of  $X$ ,  $s(A)$  will denote the set of all sequences  $(x_k)$  such that  $x_k$  is in  $A$  for  $k = 1, 2, \dots$ . In [2], a sequence  $(x_k)$  in  $X$  is called to be statistically convergent to an element  $L$  of  $X$  if for each neighbourhood  $U$  of 0,

$$\lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : x_k - L \notin U\}| = 0,$$

where the vertical bars indicate the number of elements in the enclosed set. The set of all statistically convergent sequences in  $X$  is denoted by  $st(X)$ .

Also, Çakalli [3] defined lacunary statistical convergence in topological groups as follows:

A sequence  $(x_k)$  is said to be  $S_\theta$ -convergent to  $L$  (or lacunary statistically convergent to  $L$ ) if for each neighborhood  $U$  of 0,

$$\lim_{r \rightarrow \infty} (h_r)^{-1} |k \in I_r : x_k - L \notin U| = 0.$$

In this case, we write

$$S_\theta - \lim_{k \rightarrow \infty} x_k = L \quad \text{or} \quad x_k \rightarrow L(S_\theta)$$

and define

$$S_\theta(X) = \left\{ (x_k) : \text{for some } L, S_\theta - \lim_{k \rightarrow \infty} x_k = L \right\}.$$

Now we are ready to give the main definitions of  $\mathcal{I}$ -statistical convergence and  $\mathcal{I}$ -lacunary statistical convergence in topological groups as follows:

**Definition 2.7.** A sequence  $x = (x_k)$  is said to be  $S^{\mathcal{I}}$ -statistically convergent to  $L$  or  $S^{\mathcal{I}}$ -convergent to  $L$  if for each neighborhood  $U$  of 0 and  $\delta > 0$

$$\left\{ n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : x_k - L \notin U\}| \geq \delta \right\} \in \mathcal{I}.$$

In this case we write  $x_k \rightarrow L(S^{\mathcal{I}})$ . The class of all  $S^{\mathcal{I}}$ -statically convergent sequences will be denoted by simply  $S^{\mathcal{I}}(X)$ .

*Remark 2.8.* For  $\mathcal{I} = \mathcal{I}_{fin}$ ,  $\mathcal{I}$ - statistical convergence becomes statistical convergence in topological groups which is studied by Çakalli [2].

**Definition 2.9.** A sequences  $x = (x_k)$  is said to be  $\mathcal{I}$ -lacunary statistically convergent to  $L$  or  $S_\theta^{\mathcal{I}}$ -convergent to  $L$  if for each neighborhood  $U$  of 0 and  $\delta > 0$

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} |k \in I_r : x_k - L \notin U| \geq \delta \right\} \in \mathcal{I}.$$

In this case, we write

$$S_\theta^{\mathcal{I}} - \lim_{k \rightarrow \infty} x_k = L \quad \text{or} \quad x_k \rightarrow L(S_\theta^{\mathcal{I}})$$

and define

$$S_\theta^{\mathcal{I}}(X) = \left\{ (x_k) : \text{for some } L, S_\theta^{\mathcal{I}} - \lim_{k \rightarrow \infty} x_k = L \right\}$$

and in particular,

$$S_\theta^{\mathcal{I}}(X)_0 = \left\{ (x_k) : S_\theta^{\mathcal{I}} - \lim_{k \rightarrow \infty} x_k = 0 \right\}.$$

*Remark 2.10.* For  $\mathcal{I} = \mathcal{I}_{fin}$ ,  $I$ -lacunary statistical convergence becomes lacunary statistical convergence in topological groups which is studied by Çakalli [3].

It is obvious that every  $\mathcal{I}$ -lacunary statistically convergent sequence has only one limit, that is, if a sequence is  $\mathcal{I}$ -lacunary statistically convergent to  $L_1$  and  $L_2$ , then  $L_1 = L_2$ .

### 3. INCLUSION THEOREMS

In this section, we prove the following theorems.

**Theorem 3.1.** *For any double lacunary sequence  $\theta = \{k_r\}$ ,  $S^{\mathcal{I}}(X) \subseteq S_\theta^{\mathcal{I}}(X)$  if and only if  $\liminf q_r > 1$ .*

*Proof. Sufficiency.* Suppose that  $\liminf q_r > 1$ ,  $\liminf q_r = \alpha$ , say. Write  $\beta = (\alpha - 1)/2$ . Then there exists a positive integer  $r_0$  such that  $q_r \geq 1 + \beta$  for  $r \geq r_0$ . Hence for  $r \geq r_0$ ,

$$h_r(k_r)^{-1} = (1 - (k_{r-1})(k_r)^{-1}) = (1 - (q_r)^{-1}) \geq (1 - (1 + \beta)^{-1}) = (\beta(1 + \beta_1)^{-1}).$$

Take any  $(x_k) \in S^{\mathcal{I}}(X)$  and  $S^{\mathcal{I}} - \lim_{(k) \rightarrow \infty} x_k = L$ , say. We prove that  $S_\theta^{\mathcal{I}} - \lim_{k \rightarrow \infty} x_k = L$ . Let us take any neighborhood  $U$  of 0. Then for  $r \geq r_0$ , we have

$$\begin{aligned} (k_r)^{-1} |\{k \leq k_r : x_k - L \notin U\}| &\geq (k_r)^{-1} |\{k \in I_r : x(k) - L \notin U\}| \\ &= h_r(k_r)^{-1} (h_r)^{-1} |\{k \in I_r : x_k - L \notin U\}| \\ &\geq \beta(1 + \beta)^{-1} (h_r)^{-1} \end{aligned}$$

$$|\{k \in I_r : x_k - L \notin U\}|.$$

Then for any  $\delta > 0$ , we get

$$\begin{aligned} &\{r \in \mathbb{N} : (h_r)^{-1} |\{k \in I_r : x_k - L \notin U\}| \geq \delta\} \\ &\subseteq \{r \in \mathbb{N} : (k_r)^{-1} |\{k \leq k_r : x_k - L \notin U\}| \geq \delta\beta(1 + \beta)^{-1}\} \in I. \end{aligned}$$

This proves the sufficiency.

*Necessity.* Suppose that  $\liminf_r q_r = 1$ . Then we can choose a subsequence  $\{(k_r(j))$  of the lacunary sequence  $\theta_r = \{k_r\}$  such that

$$k_{r(j)}(k_{r(j)-1})^{-1} < 1 + j^{-1}, \quad k_{r(j)-1}(k_{r(j-1)})^{-1} > j$$

where  $r(j) > r(j-1) + 2$ . Take an element  $x$  of  $X$  different from 0. Define a sequence  $(x_k)$  by  $x_k = x$  if  $k \in \{I_{r(j)}\}$  for some  $j = 1, 2, \dots, n, \dots$ , and  $x_k = 0$  otherwise. Then  $(x_k) \in S^{\mathcal{I}}(X)$  (in fact  $(x_k) \in S_0^{\mathcal{I}}(X)$ ). To see this take any neighborhood  $U$  of 0. Then we may choose a neighborhood  $W$  of 0 such that  $W \subset U$  and  $x \notin W$ . On the other hand, for each  $m$  we can find a positive number  $j_m$  such that  $k_{r(j_m)} < m \leq k_{r(j_m)+1}$ . Then

$$\begin{aligned} & (m)^{-1} |\{k \leq m : x_k \notin U\}| \leq k_{r(j_m)}^{-1} |\{k \leq m : x_k \notin W\}| \\ & \leq k_{r(j_m)}^{-1} \{|\{k \leq k_{r(j_m)} : x_k \notin W\}| + |\{k_{r(j_m)} < k \leq m : x(k) \notin W\}|\} \\ & \leq k_{r(j_m)}^{-1} |\{k \leq k_{r(j_m)} : x_k \notin W\}| + k_{r(j_m)}^{-1} (k_{r(j_m)+1} - k_{r(j_m)}) \\ & < ((j_m + 1)^{-1} + 1 + j_m^{-1} - 1) < ((j_m + 1)^{-1} + j_m^{-1}) \end{aligned}$$

for each  $m$ . Therefore  $(x_k) \in S_0^{\mathcal{I}}(X)$ . Now let us see that  $(x(k)) \notin S_0^{\mathcal{I}}(X)$ . Since  $X$  is a Hausdorff space, there exists a symmetric neighborhood  $V$  of 0 such that  $x \notin V$ . Hence

$$\begin{aligned} & \lim_{j \rightarrow \infty} (h_{r(j)})^{-1} |\{k_{r(j)-1} < k \leq k_{r(j)} : x_k \notin V\}| \\ & = \lim_{j \rightarrow \infty} (h_{r(j)})^{-1} (k_{r(j)} - k_{r(j)-1}) \\ & = \lim_{j \rightarrow \infty} (h_{r(j)})^{-1} h_{r(j)} = 1, \end{aligned}$$

and

$$\lim_{\substack{r \rightarrow \infty \\ r \neq r(j) \\ j=1,2,\dots}} h_r^{-1} |\{k_{r-1} < k \leq k_r : x_k - x \notin V\}| = 1 \neq 0.$$

Hence neither  $x$  nor 0 can be  $\mathcal{I}$ -lacunary statistical limit of  $(x_k)$ . No other point of  $X$  can be  $\mathcal{I}$ -lacunary statistical limit of the sequence as well. Thus  $(x_k) \notin S_{\theta}^{\mathcal{I}}(X)$ . This completes the proof.  $\square$

For the next result we assume that the lacunary sequence  $\theta$  satisfies the condition that for any set  $C \in F(\mathcal{I})$ ,  $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \in F(\mathcal{I})$ .

**Theorem 3.2.** *If  $\limsup_r q_r < \infty$ , then for any lacunary sequence  $\theta = \{k_r\}$ ,  $S_{\theta}^{\mathcal{I}}(X) \subseteq S^{\mathcal{I}}(X)$ .*

*Proof.* If  $\limsup_r q_r < \infty$ , then without any loss of generality, we can assume that there exists a  $0 < B < \infty$  such that  $q_r < B$  for all  $r \geq 1$ . Suppose that  $x_k \rightarrow L(S_\theta^{\mathcal{I}})$ . Take any neighborhood  $U$  of 0. For  $\epsilon, \delta, \delta_1 > 0$  define the sets

$$C = \{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : x_k - L \notin U\}| < \delta\}$$

and

$$T = \{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : x_k - L \notin U\}| < \delta_1\}.$$

It is obvious from our assumption that  $C \in F(\mathcal{I})$ , the filter associated with the ideal  $\mathcal{I}$ . Further observe that

$$A_j = \frac{1}{h_j} |\{k \in I_j : x_k - L \notin U\}| < \delta$$

for all  $j \in C$ . Let  $n \in \mathbb{N}$  be such that  $k_{r-1} < n < k_r$  for some  $r \in C$ . Now

$$\begin{aligned} \frac{1}{n} |\{k \leq n : x_k - L \notin U\}| &\leq \frac{1}{k_{r-1}} |\{k \leq k_r : x_k - L \notin U\}| \\ &= \frac{1}{k_{r-1}} |\{k \in I_1 : x_k - L \notin U\}| + \cdots + \frac{1}{k_{r-1}} |\{k \in I_r : x_k - L \notin U\}| \\ &= \frac{k_1}{k_{r-1}} \frac{1}{h_1} |\{k \in I_1 : x_k - L \notin U\}| + \frac{k_2 - k_1}{k_{r-1}} \frac{1}{h_2} |\{k \in I_2 : x_k - L \notin U\}| + \cdots + \\ &\quad + \frac{k_r - k_{r-1}}{k_{r-1}} \frac{1}{h_r} |\{k \in I_r : x_k - L \notin U\}| \\ &= \frac{k_1}{k_{r-1}} A_1 + \frac{k_2 - k_1}{k_{r-1}} A_2 + \cdots + \frac{k_r - k_{r-1}}{k_{r-1}} A_r \\ &\leq \sup_{j \in C} A_j \cdot \frac{k_r}{k_{r-1}} < B\delta. \end{aligned}$$

Choosing  $\delta_1 = \frac{\delta}{B}$  and in view of the fact that  $\bigcup\{n : k_{r-1} < n < k_r, r \in C\} \subset T$  where  $C \in F(\mathcal{I})$  it follows from our assumption on  $\theta$  that the set  $T$  also belongs to  $F(\mathcal{I})$  and this completes the proof of the theorem.  $\square$

**Problem 1.** When  $I$ -lacunary statistical convergence implies  $I$ -statistical convergence?

**Corollary 3.3.** Let  $\theta = \{(k_r)\}$  be a lacunary sequence, then  $S^{\mathcal{I}}(X) = S_\theta^{\mathcal{I}}(X)$  if and only if

$$1 < \liminf_r q_r \leq \limsup_r q_r < \infty.$$

Recall [10, 12] that an admissible ideal  $I$  is said to satisfy condition (AP) if for any mutually disjoint sequence of sets  $\{A_i\}_{i \in \mathbb{N}}$  in  $I$ , there exists a sequence  $\{B_i\}_{i \in \mathbb{N}}$  in  $I$  such that  $A_i \triangle B_i$  is finite for all  $i \in \mathbb{N}$  and  $\bigcup_{i \in \mathbb{N}} B_i \in I$ .

It was observed in [10] ( also [12] ) that for a sequence  $\{x_n\}_{n \in \mathbb{N}}$   $I$ -convergence is equivalent to  $I^*$ -convergence if and only if the ideal  $I$  satisfies the condition (AP). More facts about condition (AP) and its importance can be seen from [1]. We are now ready to prove our next result.

Of course, the  $S_\theta$ -limit is unique as shown in section 2. It is possible, however, for a sequence to have different  $S_\theta$ -limits for different  $\theta$ 's. The following theorem shows that this situation cannot occur if  $x \in S^{\mathcal{I}}(X)$ .

**Theorem 3.4.** *If  $(x(k))$  belongs to both  $S^{\mathcal{I}}(X)$  and  $S_\theta^{\mathcal{I}}(X)$ , then*

$$S^{\mathcal{I}} - \lim_{k \rightarrow \infty} x_k = S_\theta^{\mathcal{I}} - \lim_{k \rightarrow \infty} x_k.$$

*Proof.* Take any  $(x_k) \in S^{\mathcal{I}}(X) \cap S_\theta^{\mathcal{I}}(X)$  and suppose  $S^{\mathcal{I}} - \lim x = L_1$  and  $S_\theta^{\mathcal{I}} - \lim x = L_2$  and  $L_1 \neq L_2$ . Let  $0 < \epsilon < \frac{1}{2}|L_1 - L_2|$ .

Since  $X$  is a Hausdorff space, there exists a symmetric neighborhood  $U$  of 0 such that  $L_1 - L_2 \notin U$ . Then we may choose a symmetric neighborhood  $W$  of 0 such that  $W + W \subset U$ .

Since  $\mathcal{I}$  satisfies the condition (AP), so there exists  $M \in F(\mathcal{I})$  (i.e.  $\mathbb{N} \setminus M \in \mathcal{I}$ ) such that

$$\lim_{m \rightarrow \infty} \frac{1}{k_m} |\{k \leq k_m : L_1 - x_k \notin W\}| = 0, \quad \text{where } M = \{k_1, k_2, k_3, \dots\}.$$

Let  $A = \{k \leq k_m : L_1 - x_k \notin W\}$  and  $B = \{k \leq k_m : L_2 - x_k \notin W\}$ . Then  $k_m = |A \cup B| \leq |A| + |B|$ . This implies

$$1 \leq \frac{|A|}{k_m} + \frac{|B|}{k_m}. \quad \text{Since } \frac{|B|}{k_m} \leq 1 \text{ and } \lim_{m \rightarrow \infty} \frac{|A|}{k_m} = 0 \text{ so we must have } \lim_{m \rightarrow \infty} \frac{|B|}{k_m} = 1.$$

Let  $M^* = \{k_{l_1}, k_{l_2}, k_{l_3}, \dots\} = M \cap \theta \in F(\mathcal{I})$ . Then  $k_{l_p}$  <sup>th</sup> term of the statistical limit expression  $k_m^{-1} |\{k \leq k_m : L_2 - x_k \notin W\}|$ :

$$\frac{1}{k_{l_p}} |\{k \in \bigcup_{i=1}^{l_p} I_i : L_2 - x_k \notin W\}| = \frac{1}{l_p} \sum_{i=1}^{l_p} t_i h_i \quad (1)$$

$$\sum_{i=1}^{l_p} h_i$$

Where  $t_i = h_i^{-1} |\{k \in I_i : L_2 - x_k \notin W\}| \xrightarrow{\mathcal{I}} 0$  because  $x_k \rightarrow L_2(S_\theta^{\mathcal{I}})$ . Since  $\theta$  is a lacunary sequence, (1) is a regular weighted mean transform of  $t_i$ 's and therefore it is also  $\mathcal{I}$ -convergent to zero as  $p \rightarrow \infty$  and so it has a subsequence which is convergent to zero since  $\mathcal{I}$  satisfies condition (AP). But since this is a subsequence



of  $\{n^{-1}|\{k \leq n : L_2 - x_k \notin W\}\}_{n \in M}$ , we infer that  $\{n^{-1}|\{k \leq n : L_2 - x_k \notin W\}\}_{n \in M}$  is not convergent to 1 which is a contradiction. This completes the proof of the theorem.  $\square$

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