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A STUDY OF THE GROUP OF COVERING TRANSFORMATION THROUGH FUNCTORS

PRAVANJAN.KR.RANA; DEPT. OF MATHEMATICS,BERHAMPORE GIRLS'COLLEGE

Abstract. The aim of this paper is to construct a functor $Deck_x$ and we study the group of covering transformations through a functor $Deck_x$. Let $Deck_x(\tilde{X})$ denote the set of all covering transformations of a covering $p : (\tilde{X}, \tilde{x}) \to (X, x_0)$, where (X,$x_0$) is the arcwise connected, locally arcwise connected pointed topological space.

In this paper we also study the group $Deck_x(\tilde{X})$ through the fundamental group functor π_1.

In this paper we show that:

i) If $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ are equivalent covering of (X,x_0) then the groups $Deck_x(\tilde{X}_1)$ and $Deck_x(\tilde{X}_2)$ are homomorphic;

ii) $Deck_x$ is a contravariant functor;

iii) Let $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ are equivalent universal covering of (X,x_0), where X is a connected manifold, then $Deck_x(\tilde{X}_1)/f{*}\pi_1(\tilde{X}_1, \tilde{x}_1) \cong Deck_x(\tilde{X}_2)$

1. Introduction

Throughout this paper we assume that (X,x_0) is a pointed topological space and maps are base point preserving continuous surjective maps . For simplicity we write X in place of (X,x_0).

Now we recall the following definitions and statements:-

Definition 1.1. *Let $p : \tilde{X} \to X$ be a covering map. A covering transformation is a map $f : \tilde{X} \to \tilde{X}$ such that $p \circ f = p$*

Definition 1.2. *Let X and $\tilde{X}$ are connected topological spaces and $p : \tilde{X}_1 \to X$ is a covering map. $p : \tilde{X} \to X$ is called universal covering of X if it satisfies the following universal property:*

For every covering map $q : \tilde{X}_2 \to X$,with $\tilde{X}_2$ is connected and every choice of point $\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2$ with $p(\tilde{x}_1) = q(\tilde{x}_2)$,there exists exactly one fibre- preserving mapping $f : \tilde{X}_1 \to \tilde{X}_2$ such that $f(\tilde{x}_1) = \tilde{x}_2$.

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Definition 1.3. *Suppose X and $\tilde{X}$ are connected Hausdroff spaces and $p : \tilde{X} \to X$ is a covering map. The covering is called Galois if for every pair of points $\tilde{x}_0, \tilde{x}_1 \in \tilde{X}$ with $p(\tilde{x}_0) = p(\tilde{x}_1)$,there exists a covering transformation f:$\tilde{X} \to \tilde{X}$ such that $f(\tilde{x}_0) = \tilde{x}_1$.*

Definition 1.4. *Let $p_1 : (\tilde{X}_1, \tilde{x}_1) \to (X, x_0)$ and $p_2 : (\tilde{X}_2, \tilde{x}_2) \to (X, x_0)$ are two covering of (X,x_0),are called equivalent covering, if there is a homeomorphism $g : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ satisfying*
i) $g(\tilde{x}_1) = \tilde{x}_2$
ii) $p_2 \circ g = p_1$

Lemma 1.5. *Let f be a covering transformation. if $f(\tilde{x}_0) = \tilde{x}_1$,then there is at most one covering transformation f such that $f(\tilde{x}_0) = \tilde{x}_1$.*

Lemma 1.6. *Let X be a connected manifold and $p : (\tilde{X}, \tilde{x}) \to (X, x_0)$ is a universal covering, then p is a Galois and $Deck_x(\tilde{X}) \cong \pi_1(X)$.*

Lemma 1.7. *π_1 is a covariant functor from the homotopy category of pointed topological spaces to the category of groups and homomorphisms, π_1 is called the fundamental group of functor.*

Lemma 1.8. *If $f : \tilde{X} \to X$ is a Galois covering then*
i)$f * (\pi_1(\tilde{X}, \tilde{x}))$ is a normal subgroup of $\pi_1(X, x)$ and
ii)$\pi_1(X)/f * (\pi_1(\tilde{X}, \tilde{x})) \cong Deck_x(\tilde{X})$

In section 2, we construct and investigate functors and also construct the group of covering transformations of a covering and we study this group with the functors $\mathbf{Deck_x}$ and $\pi_\mathbf{1}$

2. The group of covering transformations and Functors

Let $p : (\tilde{X}, \tilde{x}) \to (X, x_0)$ be a covering and $Deck_x(\tilde{X})$ = set of all covering transformations; then we have the following **Theorems:**

Theorem 2.1. *$Deck_x(\tilde{X})$ forms a group under the composition of functions.*

Proof. It is straight forward to check that the set of deck transformations is closed under composition and the operation of taking inverses.Therefore the set $Deck_x(\tilde{X})$ forms a group. □

Theorem 2.2. *Let $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ be two coverings of (X,x_0).If $g : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ is a homeomorphism,then g induces g*: $Deck_x(\tilde{X}_2) \to Deck_x(\tilde{X}_1)$ is a group homomorphism.*

Proof. Let us define $g* : Deck_x(\tilde{X}_2) \to Deck_x(\tilde{X}_1)$ by
g*(f) $= g^{-1}fg, \forall f \in Deck_x(\tilde{X}_2)$.
Next let $h_1, h_2 \in Deck_x(\tilde{X}_2)$ be such that
$h_1 = h_2 \Rightarrow h_1 g = h_2 g \Rightarrow g^{-1}h_1 g = g^{-1}h_2 g \Rightarrow g^*(h_1) = g^*(h_2)$
g* is well defined.
Next we show that for $h_1, h_2 \in Deck_x(\tilde{X}_2)$,

$$g^*(h_1 h_2) = g^*(h_1)g^*(h_2)$$

Now

$$\begin{aligned} g^*(h_1h_2) &= g^{-1}(h_1h_2)g, by\ def. \\ &= g^{-1}(h_1gg^{-1}h_2)g \\ &= (g^{-1}h_1g)(g^{-1}h_2g) \\ &= g^*(h_1)g^*(h_2) \\ Thus\ g^*(h_1h_2) &= g^*(h_1)g^*(h_2) \end{aligned}$$

$\Rightarrow g^* is\ a\ group\ homomorphism.$

□

Corollary 2.3. *If $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ are equivalent coverings of (X,x_0),then the groups $Deck_x(\tilde{X}_1)$ and $Deck_x(\tilde{X}_2)$ are homomorphic.*

Proof. Using the definition 1.4 and the theorem 2.2,this follows. □

Proposition 2.4. *Let $p_i : (\tilde{X}_i, \tilde{x}_i) \to (X, x_0)$ be a covering maps,i=1,2,... .Then $Deck_x(\tilde{X}_i)$ and this group homomorphisms forms a category.This category is denoted by* $'$**Cgrp**$'$.

Proof. We take all the groups, of the group of covering transformations of the covering from (X,x_0) as the set of objects and the set of all covering group homomorphisms,the set of homomorphisms and the composition is the usual composition of mappings. □

Let $'$**Cov**$'$ denote the category of covering spaces of (X,x_0)and their homeomorphisms and $'$**Cgrp**$'$ denote the category of the group of covering transformations and their homomorphisms.

Then we have the following theorem:

Theorem 2.3. *For each pointed topological spaces X, there exists a contravariant functor*

$$Deck_x : Cov \to Cgrp$$

Proof. Using the theorems 2.1 and 2.2,$Deck_x(\tilde{X})$ is a group for each coverings of X and also for $g : (\tilde{X}_1, \tilde{x}_1) \to (\tilde{X}_2, \tilde{x}_2)$ is a homeomorphism in Cov,
$g^* : Deck_x(\tilde{X}_2) \to Deck_x(\tilde{X}_1)$ *by* $g^*(f) = g^{-1}fg, \forall f \in Deck_x(\tilde{X}_2)$.

Let $(\tilde{X}_1, \tilde{x}_1), (\tilde{X}_2, \tilde{x}_2)$ *and* $(\tilde{X}_3, \tilde{x}_3)$ are coverings of (X,x_0) and $\alpha : \tilde{X}_1 \to \tilde{X}_2$ and $\beta : \tilde{X}_2 \to \tilde{X}_3$ are homeomorphisms,then $\beta\alpha : \tilde{X}_1 \to \tilde{X}_3$ is also a homeomorphism.

Thus $(\beta\alpha)^* : Deck_x(\tilde{X}_3) \rightarrow Deck_x(\tilde{X}_1)$ by

$$\begin{array}{rcl} (\beta\alpha)^*(h) & = & (\beta\alpha)^{-1}h(\beta\alpha), \forall h \in Deck_x(\tilde{X}_3) \\ & = & (\alpha^{-1}\beta^{-1})h(\beta\alpha) \\ & = & \alpha^{-1}(\beta^{-1}h\beta)\alpha \\ & = & \alpha^{-1}(\beta^*(h))\alpha \\ & = & \alpha^*(\beta^*(h)) \\ & = & (\alpha^*\beta^*)(h) \end{array}$$

Thus $\forall\ h \in Deck_x(\tilde{X}_3), \quad (\beta\alpha)^* = \alpha^*\beta^*$.

Also, for identity covering transformation

$$I_{\tilde{x}} : \tilde{X} \rightarrow \tilde{X}, I^*_{\tilde{x}} : Deck_x(\tilde{X}) \rightarrow Deck_x(\tilde{X})$$

by

$$I^*_{\tilde{x}}(f) = I^{-1}_{\tilde{x}} f I_{\tilde{x}} = f = I_{Deck_x(\tilde{X})},$$

thus $Deck_x$ is a contravariant functor. □

Now using Lemma 1.6 and 1.8, we have the following Theorem:

Theorem 2.6. Let $(\tilde{X}_1, \tilde{x}_1)$ and $(\tilde{X}_2, \tilde{x}_2)$ be two equivalent coverings of (X,x_0), where X is a connected manifold, then

$$Deck_x(\tilde{X}_1)/f * \pi_1(\tilde{X}_1, \tilde{x}_1) \cong Deck_x(\tilde{X}_2)$$

Proof. Using **Definitions 1.2 , 1.4** and **Lemmas 1.6 , 1.8** and **the fundamental theorem** of homomorphism, this theorem follows. □

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Head, Department of Mathematics,
Berhampore Girl's College,
Berhampore, Murshidabad.
Pin 742102.