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APPLICATION OF ANALYTIC REPRESENTATION OF DISTRIBUTION IN THE COMPLEX ANALYSIS

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Abstract. Our aim in this paper is to present how the analytic representation of distributions can be applied in some problems in the theory of complex analysis. In fact we give two theorems in this direction.

1. Introduction

It is known that for every Schwartz distribution $T \in D'$ there exists a complex function f, analytic on the complex plane C unless some closed subset of R, whish is the support of T and

$$f\,(x+iy) - f\,(x-iy) \to T \ \text{ in } \ D' \text{ as } \ y \to 0^{+}.$$

The function f is called analytic representation of the distribution T. The analytic representation is determined up to an entire function. Of course every entire function is the analytic representation of the zero distribution ([1]).

A class of distributions whose analytic representation is determined by the Cauchy kernel function

$$t \to \frac{1}{2\pi i\,(t-z)},$$

is very well known

$$\hat{T}(z) = \frac{1}{2\pi i} < T_t, \frac{1}{t-z} >, z \in C \setminus \operatorname{supp} T \tag{1}$$

and in that case the analytic representation is also called a Cauchy representation of the given distribution ([1]).

Key words and phrases. distrubution, analytic representation.

Here we will consider analytic functions defined by integrals as, for example, is the Gamma function

$$\Gamma(z) = \int_0^\infty t^{z-1} \exp(-t)\, dt \text{ for } \operatorname{Re} z > 0$$

which can be analyticly extended on the domain $\Omega = C \backslash (-Z_+)$ where $Z_+ = \{0, 1, \dots\}$ by the formula $\Gamma(z) = \dfrac{\Gamma(z+m)}{z(z+1)\dots(z+m-1)}$. In the point $-m, m \in Z_+$, Γ has a simple pole with $\operatorname{Res}(\Gamma, -m) = \dfrac{(-1)^m}{m!}$.

The Gamma function is the analytic representation of the following distribution $T = 2\pi i \sum\limits_{m=0}^{\infty} \dfrac{(-1)^{m+1}}{m!} \delta_{-m}$ where δ_{-m} is the Dirac distribution at the point $-m$ ([1]).

It is clear that $\operatorname{supp} T = \{0, -1, \dots, -m, \dots\}$ i.e. it consits of the points in which the Gamma function has the singularities.

2. Two theorems concerning application of analytic representation of distribution in the complex analysis

The fact that a given function is the analytic representation for some distribution T can be of interest in the study of this function with the aid of the properties of the analytic representation. For example, to determine the domain of analyticity, or the jump as z crosses through any point $x \in R$, from the upper half plane to the lower half plane of the complex plane C.

Let us consider

$$g(z) = \int_0^\infty \frac{dt}{\exp t - z}, \quad z \in C \setminus [1, \infty)$$

We will determine the domain of analyticity and the jump as z crosses the real line at the point x from the upper half plane Π^+ to the lower half plane Π^-.

Of course, this problem is the ordinary one in the complex analysis, but we will consider it by the means of the analytic representation.

To do this, we first make the change $u = \exp t$ and obtain

$$g(z) = \int_1^\infty \frac{\frac{1}{u}du}{u-z}, \quad z \in C \setminus [1,\infty).$$

Now we consider the function $u \to \frac{1}{u}H(u-1)$, where H is the Heaviside function, as a regular distribution T whose supp$T = [1,\infty)$. This distribution has a Cauchy representation

$$\hat{T}(z) = \frac{1}{2\pi i}\int_1^\infty \frac{\frac{1}{u}H(u-1)}{u-z}du, \quad z \in C \setminus [1,\infty)$$

Since supp$T = [1,\infty)$ from the analytic representation we conclude that the function $\hat{T}$ is analytic in the domain $\Omega = C \setminus [1,\infty)$.

Since $\hat{T}(x+iy) - \hat{T}(x-iy) \to \frac{1}{x}H(x-1)$, as $y \to 0^+$ in the distribution sense, this fact suggest us that $\frac{1}{x}H(x-1)$ is the jump of $\hat{T}$ as z crosses $x \in (1,\infty)$ and then it is easy to verify this. Since $g = 2\pi i\,\hat{T}$, we can conclude that the function g is analytic function on the domain $\Omega = C \setminus [1,\infty)$ and that in $x \in (1,\infty)$, it has jump $\frac{2\pi i}{x}$ and the jump is 0 at $x < 1$.

In the following part we present, in the form of theorems, a more complicated cases.

Theorem 2.1. *If f and g are continuous real valued functions defined on an interval $[a,b]$ and if $f' > 0$ is also continuous on (a,b), then*

$$F(z) = \int_a^b \frac{g(t)\,dt}{f(t)-z} \tag{2}$$

is analytic in $C\setminus[c,d]$, $c = f(a)$, $d = f(b)$ and the jump is $g\left(f^{-1}(x)\right)\left(f^{-1}(x)\right)'$, $x \in [c,d]$.

Proof. From the integral which defines the function, it is not clear where this function is analytic and what is its jump at $x \in R$. We will use the analytic representation, and in order to do so we first, make a change of variable , and give the function in the form

$$F(z) = \int_{c}^{d} \frac{g\left(f^{-1}(t)\right)\left(f^{-1}(t)\right)'}{t-z} dt$$

where $c = f(a)$ and $d = f(b)$.

Now we consider the function

$$h\,(t) = g\left(f^{-1}\,(t)\right)(f^{-1}\,(t))' H\,(t-c)\,H\,(d-t)\,. \tag{3}$$

This function determines a regular distribution with support $[c, d]$ and thus has the Cauchy representation

$$\hat{h}\,(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g\left(f^{-1}\,(t)\right)(f^{-1}\,(t))' H\,(t-c)\,H\,(d-t)}{t-z} dt.$$

The function $\hat{h}\,(z)$ as a Cauchy representation is analytic on the domain $C \setminus [c, d]$. Further, since $\hat{h}\,(x+iy) - \hat{h}\,(x-iy) \to h\,(x)$ as $y \to 0^{+}$ in distribution sense, this suggest us that the jump of $\hat{h}\,(z)$ at $x \in (c, d)$ is equal to $g\left(f^{-1}\,(x)\right)(f^{-1}\,(x))'$, which is not difficult to prove. Now since $F(z) = 2\pi i\,\hat{h}(z)$, we conclude that $F(z)$ is analytic on $C \setminus [c, d]$ with the jump $2\pi i g\left(f^{-1}\,(x)\right)(f^{-1}\,(x))'$ for $x \in (c, d)$ and zero for $x < c$.

Theorem 2.2. *Let $g(t)$ be a continuous function on the interval (r, ∞), $r > 0$ and $g(t) = 0$ for $t \leq r$ and let $g(t)$ be a locally integrable which satisfies $g\,(t) = O\left(|t|^{\beta}\right)$ as $|t| \to \infty$ for $\beta < 0$. Then the function*

$$\hat{g}\,(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g\,(t)}{t-z} dt, Imz \neq 0 \tag{4}$$

is analytic in the domain $\Omega = C \setminus [r, \infty)$ and is an analytic continuation of a power series whose coefficients are determined by the function g.

Proof. The function g with the given properties is a regular distribution which has the Cauchy representation given by the formula (4). After the change of variable we have that

$$\hat{g}\,(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{g\,(\exp t) \exp t}{\exp t - z} dt = \frac{1}{2\pi i} \int_{\ln r}^{\infty} \frac{g\,(\exp t)}{1 - z \exp\,(-t)} dt.$$

For $|z| < r$ we have

$$\hat{g}(z) = \frac{1}{2\pi i} \int_{\ln r}^{\infty} g(\exp t) \sum_{n=0}^{\infty} z^n \exp(-nt) dt = \sum_{n=0}^{\infty} a_n z^n$$

where

$$a_n = \frac{1}{2\pi i} \int_{\ln r}^{\infty} g(\exp t) \exp(-nt)\, dt.$$

Since $\hat{g}(z)$ is an analytic function in the domain $\Omega = C \setminus [r, \infty)$ it follows that the radius of convergence of the power series is equal to r and in the point r has a singularity. The function $\hat{g}(z)$ is analytic extension in the domain Ω for the power series.

Of particular interest are the functions of the form $(t-a)^{\beta} H(t-a)$ for $-1 < \beta < 0$ which are regular distributions and whose derivatives are the distributions $\beta PV(t-a)^{\beta-1} H(t-a)$ where PV is the principal value in the sense of Adamard.

Note. In some cases we have the opposite task: to determine the analytic continuation of a given power series.

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