CERTAIN INTEGRALS INVOLVING IN THE STUDY OF JACOBI AND LAGUERRE POLYNOMIALS

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Dedicated to Professor Akitsugu Kawaguchi on his 80th birthday

1. The method of Christoffel-Darboux for the Jacobi polynomials defined by standard relation in terms of an ordinary hypergeometric function [1][1)]

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}{}_2F_1(-n,\ n+\alpha+\beta+1;\ \alpha+1;\ (1-x)/2)\,,$$

with $\mathrm{Re}\,\alpha>-1$, $\mathrm{Re}\,\beta>-1$, $-1\leq x\leq 1$, leads us to the identity

$$\sum_{\kappa=0}^{n} P_\kappa^{(\alpha,\beta)}(x)P_\kappa^{(\alpha,\beta)}(y)/h_\kappa^{(\alpha,\beta)}=\frac{2^{-\alpha-\beta}}{2n+\alpha+\beta+2}\,\frac{\Gamma(n+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}$$

$$\times\frac{P_{n+1}^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(y)-P_n^{(\alpha,\beta)}(x)P_{n+1}^{(\alpha,\beta)}(y)}{x-y}$$

$$=K_n^{(\alpha,\beta)}(x,\ y)\,,$$

where

$$h_\kappa^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}}{2\kappa+\alpha+\beta+1}\,\frac{\Gamma(\kappa+\alpha+1)\Gamma(\kappa+\beta+1)}{\Gamma(\kappa+1)\Gamma(\kappa+\alpha+\beta+1)}\,,\qquad x\neq y\,.$$

Let $f(x)$ is an arbitrary polynomial given by

$$f(x)=\sum_{k=0}^{n} a_\kappa P_\kappa^{(\alpha,\beta)}(x)\,.$$

Then we have

$$(1)\qquad \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta K_n^{(\alpha,\beta)}(x,\ y)f(x)dx=f(y)\,.$$

2. If we put $y=1$, since

$$K_n^{(\alpha,\beta)}(x,\ 1)=K_n^{(\alpha,\beta)}(x)$$

$$=2^{-\alpha-\beta-1}\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(n+\beta+1)}\,P_n^{(\alpha+1,\beta)}(x)\,,$$

from (1) we obtain

$$(2)\qquad \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha+1,\beta)}f(x)dx=2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+2)}\,f(1)$$

$$=2^{\alpha+\beta+1}B(\alpha+1,\ n+\beta+1)f(1)\,.$$

Similarly, because $P_n^{(\alpha,\beta)}(-1)=(-1)^n(1+\beta)_n/n!$, we get

$$(3)\qquad \int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta+1)}(x)f(x)dx=(-1)^n 2^{\alpha+\beta+1}B(\beta+1,\ n+\alpha+1)f(-1)\,.$$

Using the relation $P_n^{(\alpha,\beta-1)}(x)-P_n^{(\alpha-1,\beta)}(x)=P_{n-1}^{(\alpha,\beta)}(x)$, (2) and (3) yield

$$(4)\qquad \int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta-1}P_n^{(\alpha,\beta)}(x)f(x)dx=2^{\alpha+\beta-1}[B(\alpha,\ n+\beta+1)f(1)$$

$$+(-1)^n B(\beta,\ n+\alpha+1)f(-1)]\,.$$

1) Numbers in brackets refer to the references at the end of the paper.

3. Particular cases. If we take $f(x)=P_k^{(\gamma,\delta)}$, γ, $\delta>-1$, we obtain from (2)

$$\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha+1,\beta)}(x)P_\kappa^{(\gamma,\delta)}(x)dx=2^{\alpha+\beta+1}\binom{\gamma+\kappa}{\kappa}B(\alpha+1,\,n+\beta+1)\,,\qquad \kappa\leq n\,,$$

and from (4) it follows

$$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta-1}P_n^{(\alpha,\beta)}(x)P_\kappa^{(\gamma,\delta)}(x)dx=2^{\alpha+\beta-1}\Big[\binom{\gamma-\kappa}{\kappa}B(\alpha,\,n+\beta+1)$$

$$+(-1)^{n+\kappa}\binom{\delta+\kappa}{\kappa}B(\beta,\,n+\alpha+1)\Big],$$

which by $\gamma=\delta=0$ becomes

$$\int_{-1}^{1}(1-x)^{\alpha-1}(1+x)^{\beta-1}P_n^{(\alpha,\beta)}(x)P_\kappa(x)dx=2^{\alpha+\beta-1}[B(\alpha,\,n+\beta+1)+(-1)^{n+\kappa}B(\beta,\,n+\alpha+1)]\,,$$

where $P_\kappa(x)$ is Legendre polynomial.

4. *The "Kernel polynomial"* $K_n^{(a)}(x,\,y)$ for Laguerre polynomials $L_n^{(a)}(x)$ is given by [3]

$$\Gamma(a+1)K_n^{(a)}(x,\,y)=\sum_{k=0}^{n}\frac{\kappa!L_\kappa^{(a)}(x)L_\kappa^{(a)}(y)}{(1+a)_k}=\frac{(n+1)!}{(a+1)_n}\,\frac{L_n^{(a)}(x)L_{n+1}^{(a)}(y)-L_{n+1}^{(a)}(x)L_n^{(a)}(y)}{x-y}\,,$$

where $L_n^{(a)}(x)=((\alpha+1)_n/n!)\;{}_1F_1(-n,\,\alpha+1;\,x)$, and $x\neq y$.

Taking one polynomial of the form $\phi(x)=\sum_{s=0}^{n}a_sL_s^{(a)}(x)$, we obtain

$$\int_0^\infty e^{-x}x^aK_n^{(a)}(x,\,y)\phi(x)dx=\phi(y)\,. \tag{6}$$

In particular, for $y=0$, since $L_n^{(a)}(0)=(a+1)_n/n!$ and $\Gamma(a+1)K_n^{(a)}(x,\,0)=L_n^{(a+1)}(x)$, from (6) we have

$$\int_0^\infty e^{-x}x^aL_n^{(a+1)}(x)\phi(x)dx=\Gamma(a+1)\phi(0)\,. \tag{6'}$$

Special case:

$$1°.\quad\int_0^\infty e^{-x}x^aL_n^{(a+1)}(x)L_k^{(b)}(x)dx=(b+1)_k\Gamma(a+1)/k!\,, \tag{7}$$

$$2°.\quad\int_0^\infty e^{-x}x^aL_n^{(a+1)}(x)P_{2k}(x)dx=(-1)^k\left(\frac{1}{2}\right)_k\Gamma(a+1)/k!\,, \tag{8}$$

$$3°.\quad\int_0^\infty e^{-x}x^aL_n^{(a+1)}(x)H_{2k}(x)dx=(-1)^k2^k(2k-1)!!\Gamma(a+1)\,, \tag{9}$$

where $H_n(x)$ is Hermite polynomial.

5. By the relation $L_n^{(a+b+1)}(x+y)=\sum_{k=0}^{n}L_k^{(a)}(x)L_{n-k}^{(b)}(y)$, the formula (6)′ yields

$$\int_0^\infty e^{-x}x^{a-1}L_n^{(a)}(x)L_k^{(a+b)}(x+y)dx=\Gamma(a)L_k^{(a+b)}(y)$$

$$=\Gamma(a)\sum_{r=0}^{k}\frac{(a)_r}{r!}\,L_{k-r}^{(b-1)}(y)$$

and

$$\int_0^\infty e^{-x}x^{a-1}L_n^{(a)}(x)L_k^{(b+1)}(x)dx=\Gamma(a)\sum_{r=0}^{k}\frac{(a)_r(b-a)_{k-r}}{r!\,(k-r)!}\,. \tag{10}$$

Comparing (7) and (10) we find the identity

$$\sum_{r=0}^{k}\binom{k}{r}(b-a)_{k-r}(a)_r=(b)_k\,.$$

REFERENCES

[1] G. Szegö: Orthogonal polynomials, *Coll. Publications*, V. xxiii, (1959).

[2] M. Parodi: *C. R.* (Serie A), **268** (1969), 1185.

[3] E. Rainville: Special functions, *New York*, (1960).