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DISTRIBUTIONNAL BOUNDARY VALUES OF AN INNER FUNCTION

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Abstract

In this paper we find distributional boundary values of an inner functions in the upper half plane.

We denote by Π^+ the upper half plane i.e. $\Pi^+ = z\colon I_m z > 0\}$. If f is holomorphic in Π^+ and bounded, thus is $|f(z)| \leq C$, for every $z \in \Pi^+$, then boundary value function

$$f^*(x) = \lim_{y \to 0^+} f(x+iy)$$

exist almost everywhere an $\mathbf{R}$ and $f^* \in L^\infty(\mathbf{R})$. We call the homeorphic function $I(z)$, $z \in \Pi^+$ an inner function if $|I(z)| < 1$, and it's boundary function $I^*(x)$ satisfy $|I^*(x)| = 1$ a.e. on $\mathbf{R}$.

Let $S(\mathbf{R})$ be the class of rapidly decreasing infinitly differentiable functions $f \in H^P(C \setminus R)$ and $g \in S(\mathbf{R})$, then the limit

$$T_f(g) = \lim_{y \to 0} \int_{\mathbf{R}} \big[f(x+iy) - f(x-iy)\big] g(x)\, dx$$

exists and is a tempered distribution

The limit

$$f_g^*(x) = \lim_{y \to 0'} \int_{\mathbf{R}} f(x+iy) g(x)\, dx$$

exists a.e. and is called distributional boundary value in the sence of $S(\mathbf{R})$.

Following [3] we use the following form of the inner function in the upper half plane

$$I(z) = \exp\Big(\frac{1}{i}\int_{\mathbf{R}} \frac{1+tz}{z-t}\,d\mu(t)\Big)$$

where $\mu(t)$ is a positive singular mesure whith respect to the Lebesgue measure on $\mathbf{R}$ and

$$\int_{\mathbf{R}} (1+t^2)d\mu(t) < \infty$$

In our case, let $\mu(t)$ be the point mass, that is $\mu(\mathbf{R}) = t_0,\ t_0 > 0$. Put

$$\varphi(t_0, z) = \frac{1}{i}\,\frac{1+t_0 z}{z-t_0}\,t_0 \quad t_0 > 0\,.$$

Theorem. *Let*

$$I(z) = \exp\varphi(t_0, z) \quad t_0 > 0\,.$$

Then if $g \in S(\mathbf{R})$,

$$\begin{aligned} T_I(g) &= \lim_{y\to 0^+}\int_{\mathbf{R}} \big(I(x+iy) - I(x-iy)\big)g(x)\,dx \\ &= -2(1+t_0^2)t_0\pi g(t_0)\,. \end{aligned}$$

Proof. Note that

$$\varphi(t_0, z) = -\frac{y(1+t_0^2)t_0}{(x-t_0)^2+y^2} - i\,\frac{(1+t_0x)(x-t_0)+t_0y^2}{(x-t_0)^2+y^2}\,t_0$$

Using the series development of the exponential function, we have

$$I(x+iy) - I(x-iy) = -\frac{2y(1+t_0^2 t_0}{(x-t_0)^2+y^2} + J(x,y,t_0)$$

where

$$\begin{aligned} J(x,y,t_0) =& \frac{1}{2!}\left[\varphi^2(t_0,z) - \varphi^2(t_0\overline{z})\right] + \cdots \\ &+ \frac{1}{n!}\left[\varphi^n(t_0,z) - \varphi^n(t_0,\overline{z})\right] + \cdots\,. \end{aligned}$$

Elementary estimations shows that

$$\left|\varphi^n(t_0,z)-\varphi^n(t_0,\overline{z}\right| \le \frac{2y(1+t_0^2)t_0 n\left|\varphi^{n-1}(t_0,z)\right|}{(x-t_0)^2+y^2}\,.$$

Let $g \in S(\mathbf{R})$. Since g is a bounded and the convergence of the series is uniform on the compact subsets of the plane, we have

$$\lim_{y\to\infty}\lim_{M\to\infty}\int_{-M}^{M} J(x,y,t_0)\,dx = 0\,.$$

Using the Tayler series develompent od the function $p \in S(\mathbf{R})$ around $t = t_0$, we have

$$\begin{aligned} g(x) &= g(t_0) + g'(t_0)(x-t_0) + \frac{g''(t_0)}{2!}(x-t_0)^2 + \cdots \\ &= g(t_0) + h(x,t_0)(x-t_0) \end{aligned}$$

where h is bounded. We have

$$\lim_{y\to 0^+}\lim_{M\to\infty}\int_{-M}^{M}\big(I(x+iy)-I(x-iy)\big)g(x)dx =$$

$$= \lim_{y\to 0^+}\lim_{M\to\infty}\int_{-M}^{M}\frac{-2y(1+t_0^2)t_0}{(x-t_0)^2+y^2}\,g(x)\,dx +$$

$$+ \lim_{y\to 0^+}\lim_{M\to\infty}\int_{-M}^{M} J(x,y,t_0)g(x)dx$$

$$= -\,2\pi(1+t_0^2)t_0\,g(t_0)\,.$$

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ДИСТРИБУЦИОНИ ГРАНИЧНИ ВРЕДНОСТИ НА ЕДНА ВНАТРЕШНА ФУНКЦИЈА

Никола Пандески

Р е з и м е

Во оваа работа докажавме дека за секундарната внатрешна функција

$$I(z) = \exp\left(\frac{1+t_0 z}{t-t_0}\,t_0\right), \quad t_0 > 0$$

дека ако $g \in S(\mathbf{R})$, тогаш

$$T_I(g) = -2\pi(1+t_0^2)t_0\, g(t_0)\,.$$

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