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ON A MEAN VALUE ON INTERVAL $[a,b]$ IN THE CONTEXT OF COMPLEMENTARY AND RECIPROCAL MEANS

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Abstract

In introduction - statement 1°, besides "parallelogram of the means" from article [1], we also state the means (5) from the article [2]. In the statement 2°, we give the features of the means (5) on the graph of the function $y = M_x(a,b)$ together with calculating their corresponding indexes on the graph. In the statement 3°, the mean $K[\overline{M}(a,b)]$ is considered, as well as its reciprocal mean.

1⁰. In the paper [1] by figure 2 there are the graphs of the functions given

$$y = M_x(a,b) \quad \text{and} \quad y_1 = R[M_x(a,b)] = M_{1-x}(a,b)\,, \tag{1}$$

where the graph of the reciprocal function $R[M_x(a,b)]$ is axis symmetrical to the graph of the function $M_x(a,b)$ in relation to the line $x = \frac{1}{2}(R[M_{\frac{1}{2}+x}(a,b)]) = M_{\frac{1}{2}-x}(a,b))$ and to the middle points marked on them:

$$H\Big(0, \frac{2ab}{a+b}\Big), \quad G\Big(\frac{1}{2}, \sqrt{ab}\Big), \quad A\Big(1, \frac{a+b}{2}\Big),$$
$$L\Big(\frac{3}{2}, \frac{a^{\frac{3}{2}} + b^{\frac{3}{2}}}{a^{\frac{1}{2}} + b^{\frac{1}{2}}}\Big), \quad M\Big(2, \frac{a^2+b^2}{a+b}\Big), \tag{2}$$

where the quadrable $HGML$ is a parallelogram. Namely, from the complementarity:

$$M(a,b) + H(a,b) = G(a,b) + L(a,b) = a + b, \tag{3}$$

follows

$$\begin{aligned} &G(a,b) - H(a,b) = M(a,b) - L(a,b) \quad \text{and} \\ &L(a,b) - H(a,b) = M(a,b) = M(a,b) - G(a,b). \end{aligned} \tag{4}$$

Next, in the paper [2] the mean is introduced, which is marked here as $\overline{M}(a,b)$:

$$\overline{M}(a,b) = \frac{4ab}{a^2 + 6ab + b^2} \quad \text{and} \quad R[\overline{M}(a,b)] = \frac{A(a,b) + H(a,b)}{2}. \tag{5}$$

2⁰. At this point we give the graphic construction of the means $\overline{M}(a,b)$ and $R[\overline{M}(a,b)]$ on the graph of the function $y = M_x(a,b)$, as well as the verification of the relations:

$$G(a,b) < R[\overline{M}(a,b)] < A(a,b) \quad \text{and} \quad H(a,b) < \overline{M}(a,b) < G(a,b); \tag{6}$$

and the calculation of their corresponding indexes on the graph of the function.

According to the downward convexity of the graph of the function $y = M_x(a,b)$ on the interval [0,1] is true (figure):

$$\begin{aligned} &R[\overline{M}(a,b)] = \frac{A(a,b) + H(a,b)}{2} > G(a,b) \quad \text{and} \\ &R[\overline{M}(a,b)] < A(a,b), \end{aligned} \tag{7}$$

i.e.

$$R[\overline{M}(a,b)] \in (G(a,b),\, A(a,b)) \;-\; \text{the point } \overline{A}. \tag{8}$$

Next, taking into account thant $y = M_x(a,b)$ strictly increases

$$R[\overline{M}(a,b)] \in M_x(a,b) \tag{9}$$

for the unique $x = x_0 \in \left(\frac{1}{2}, 1\right)$ – the point B on the graph of the function $y = M_x(a,b)$ was created by the intersection of the line from point $\overline{A}$, being parallel to x-axis and the mentioned graph.

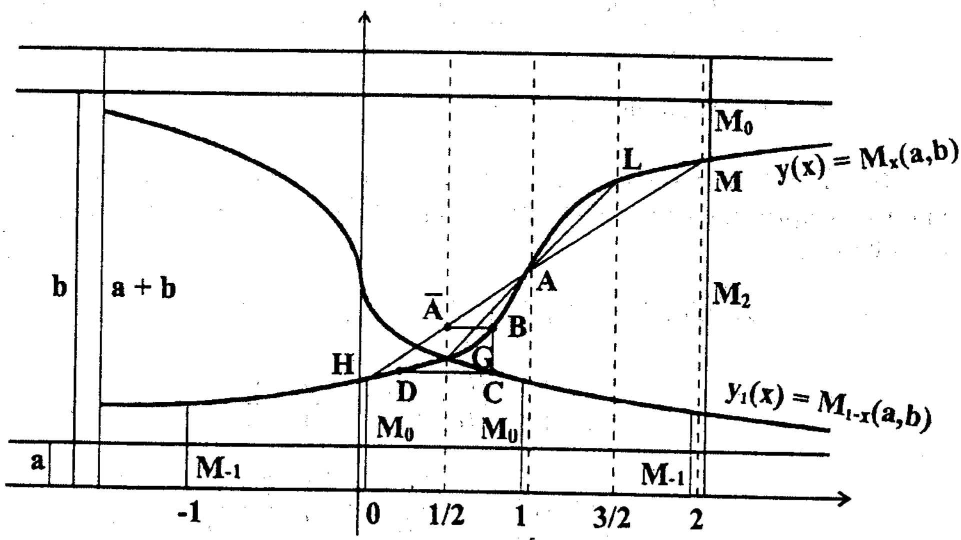
Next, the perpendicular from the point B to x-axis in the intersection with the graph of the function $R[M_x(a,b)] = M_{1-x}(a,b)$, brings us to the point C on the graph of the function $y_1 = M_{1-x}(a,b)$. For $x_0 = \frac{1}{2} + t_0$, $t_0 \in \left(0, \frac{1}{2}\right)$, we obtain that [1]:

$$R[M_{\frac{1}{2}+t_0}(a,b)] = M_{\frac{1}{2}-t_0} = M_{\overline{x}}(a,b)\,, \tag{10}$$

where $\overline{x} = \frac{1}{2} - t_0 \in \left(0, \frac{1}{2}\right)$. Thus, the perpendicular from the point C onto y-axis in the intersection with the graph of the function $y = M_x(a,b)$, brings us to the value $\overline{M}(a,b)$ where

$$\overline{M} = M_{\overline{x}}(a,b),\ \ \overline{x} \in \left(0, \frac{1}{2}\right) \quad \text{and} \quad H(a,b) < M_{\overline{x}}(a,b) < G(a,b) \tag{11}$$

– point D on the graph of function $y = M_x(a,b)$ (figure).



Figure

We are going to determine the numerical value for $\overline{x}$. The function

$$y = \frac{a^x + b^x}{a^{x-1} + b^{x-1}} \tag{12}$$

is an increasing one, so it has the inverse function which is also increasing. From (12) it follows that

$$\left(\frac{a}{b}\right)^{1-x} = \frac{b-y}{y-a}. \tag{13}$$

In our case

$$y = \frac{4ab(a+b)}{a^2 + 6ab + b^2}, \tag{14}$$

according to the relation (13), we obtain that

$$\overline{x} = \frac{\log(a+3b) - \log(b+3a)}{\log b - \log a}, \tag{15}$$

where $\overline{x} \in \left(0, \frac{1}{2}\right)$. Namely, the following inequality is true:

$$0 < \frac{\log(a+3b) - \log(b+3a)}{\log b - \log a} < \frac{1}{2}, \quad b > a > 0. \tag{16}$$

Index x_0 is determined by the relation: $x_0 = 1 - \overline{x}$.

3^0. To estimate the complementary mean value for the mean $\overline{M}(a,b)$, and the reciprocal mean value for the mean $K[\overline{M}(a,b)]$ there is the following calculation:

$$\begin{aligned}
K[\overline{M}(a,b)] &= K[H(A,H)] \\
&= K\left[\frac{2AH}{A+H}\right] = 2A - \frac{2AH}{A+H} \\
&= 2A\left(1 - \frac{H}{A+H}\right) = \frac{A}{H}\,\frac{2AH}{A+H} \\
&= \frac{A}{H}\,H(A,H) = \frac{A}{H}\,\overline{M}(a,b),
\end{aligned}$$

and

$$\begin{aligned} R[K[\overline{M}(a,b)]] &= R\Big[\frac{A}{H}\,\overline{M}(a,b)\Big] \\ &= \frac{H}{A}\,R[\overline{M}(a,b)] = \frac{H}{A}\,A(A,H) \\ &= \frac{H}{A}\,\overline{A}(a,b)\,, \end{aligned}$$

where $\overline{A}(a,b) = A(A,H)$. Finally

$$K[\overline{M}(a,b)] = \frac{A}{H}\,\overline{M}(a,b) \quad \text{and} \quad R[K[\overline{M}(a,b)]] = \frac{H}{A}\,\overline{A}(a,b)\,. \tag{17}$$

References

[1] J. V. Malešević: *On a mean value on the interval* $[a,b]$, *classic mean values and geometric interpretation,* Glasnik Šumarskog fakulteta, Beograd, 1996-1997, No. 78-79, pg. 79-90. (in Serbian)

[2] J. V. Malešević: *On an inequalty and a mean value,* RGMIA Research Reports Collection, Vol. 3. No. 2, 2000, pg. 281-287.

ЗА ЕДНА СРЕДНА ВРЕДНОСТ НА СЕГМЕНТОТ $[a, b]$ ВО КОНТЕКСТ НА КОМПЛЕМЕНТАРНИ И РЕЦИПРОЧНИ СРЕДИНИ

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Резиме

Во воведниот дел се наведува, "паралелограм на средини" од трудот [1], и средини (5) од трудот [2]. Дадени се одредбите на срединатa (5) на графикот на функцијата $y = M_x(a, b)$, со пресметување на нивните кореспондентни индекси на тој график. Во контекст на насловот интерпретирана е и срединатa $K[M(a, b)]$.

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