A NOTE ON ORTHOGONAL POLYNOMIALS

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This is an extension of the results of M. Parodi [1] and the author [2] concerning the Legendre and Jacobi polynomials. We construct a class of polynomials belonging to the vector space V of polynomials of degree less than or equal to n, whose basis are the orthogonal polynomials $p_n(x)$. They have a given common zero $\omega \varepsilon R$ or C. The other zeros are in the interval (a, b) which contains the zeros of $p_n(x)$. In fact, we show that theise polynomials are conected with kernel polynomials.

1. Let $p_n(x)$, $n = 1, 2, \ldots$ be a set of orthogonal polynomials. It is known that for any three consecutive polynomials one has a relation [3]

$$p_n(x) = (a_n x + b_n) p_{n-1}(x) - c_n p_{n-2}(x), \quad n \geqslant 1,$$

where a_n, b_n and c_n are constants with $a_n > 0$ and $p_{-1}(x) = 0$.

Consider the polynomial $R_n(x, \lambda, \omega)$ of degree n $(n>2)$ defined by

$$R_n(x,\lambda,\omega) = \begin{vmatrix} a_n(x-\omega) + K_n c_n & -K_n c_n & 0 & \cdots & 0 & 0 \\ a_{n-1}(\lambda-\omega) - 1/K_n & a_{n-1}x + b_{n-1} & -K_{n-1} c_{n-1} & \cdots & 0 & 0 \\ a_{n-2}(\lambda-\omega) & -1/K_{n-1} & a_{n-2}x + b_{n-2} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & & \\ a_2(\lambda-\omega) & 0 & 0 & \cdots & a_2 x + b_2 & -K_2 c_2 \\ a_1(\lambda-\omega) & 0 & 0 & \cdots & -1/K_2 & a_1 x + b_1 \end{vmatrix}$$

with

$$a_n\lambda + b_n - K_n c_n = 1/K_{n+1}, \qquad n = 2, 3, \ldots, n-1,$$
$$a_1\lambda + b_1 = 1/K_2$$

Expanding the determinant we find the expresion of the polynomial in the form

$$\begin{aligned} R_n(x, \lambda, \omega) = {} & p_n(x) - p_{n-1}(x)/K_{n+1} + (\lambda-\omega)\,[a_n p_{n-1}(x) + \\ & + a_{n-1} K_n c_n p_{n-2}(x) + a_{n-2} K_n K_{n-1} c_n c_{n-1} p_{n-3}(x) + \ldots + \\ & + a_2 K_n \ldots K_2 c_n \ldots c_2 p_0(x)] \end{aligned} \tag{1}$$

If we add to the elements of the first columns the elements of the others columns until the n-th, we remark that the elements of the first column of the transformed determinant have the factor $x-\omega$ in common, which leads to the relation.

$$\begin{aligned} R_n(x, \lambda, \omega) = {} & (x-\omega)\,[a_n p_{n-1}(x) + a_{n-1} K_n c_n p_{n-2}(x) + \\ & + a_{n-2} K_n K_{n-1} c_n c_{n-1} p_{n-3}(x) + \ldots + a_1 K_n \ldots K_2 c_n \ldots c_2 p_0(x)] \end{aligned} \tag{2}$$

Taking $\omega = \lambda$, from (1) and (2) it follows that

$$\begin{aligned} p_n(x) - p_{n-1}(x)/K_{n+1} = {} & (x-\lambda)\,[a_n p_{n-1}(x) + a_{n-1} K_n c_n p_{n-2}(x) + \\ & + \ldots + a_1 K_n \ldots K_2 c_n \ldots c_2 p_0(x)] \end{aligned}$$

The zeros of the polynomial $p_n(x) - p_{n-1}(x)/K_{n+1}$ lie in the open interval (a, b). Then the zeros of the polynomial $R_n(x, \lambda, \omega)$ distinct from ω, lie in the same interval (a, b).

One transformation of the preceding relation yields for the polynomial $R_n(x, \lambda, \omega)$ the following form

$$R_n(x,\lambda,\omega) = \frac{x-\omega}{x-\lambda}(p_n(x) - p_{n-1}(x)/K_{n+1})$$

2. Put

$$\frac{p_{r-2}(\lambda)}{p_{r-1}(\lambda)} = K_r, \qquad r = 1, 2, \ldots n, n+1 .$$

The kernel polynomial $Q_n(x, \lambda)$ given by

$$Q_n(x, \lambda) = a_n \sum_{k=0}^{n-1} p_k(\lambda)\, p_k(x),$$

since $a_k = c_k\, a_{k-1}$, $k = 1, 2, \ldots n$, has the form

$$Q_n(x, \lambda) = a_n p_{n-1} + K_n a_{n-1} c_n p_{n-1}(x) + \ldots +$$
$$+ a_2 K_n \ldots K_2 c_n \ldots c_2 p_0(x). \qquad (3)$$

Comparing (2) and (3) we obtain

$$R_n(x, \lambda, \omega) = (x - \omega)\, Q_n(x, \lambda)$$

3. *Special cases.*

a) Jacobi polynomials. In this case we have

$$a_n = (\alpha + \beta + 2n - 1)(\alpha + \beta + 2n) / 2n(\alpha + \beta + n)$$

$$b_n = (\alpha^2 - \beta^2)(\alpha + \beta + 2n + 1) / 2n(\alpha + \beta + n)(\alpha + \beta + 2n - 2)$$

$$c_n = (\alpha + \beta + 2n)(\alpha + n - 1)(\beta + n - 1)/n(\alpha + \beta + n)(\alpha + \beta + 2n - 2)$$

Taking $\lambda = 1$, Since

$$\frac{p_n^{(\alpha,\beta)}(x) - \frac{\alpha + n}{n} p_{n-1}(x)}{x - 1} = \frac{\alpha + \beta + 2n}{2n} \frac{(\beta)_n}{(\alpha + \beta + 1)_n}$$

$$\sum_{k=1}^{n} (\alpha + \beta + 2n - 2k + 1) \frac{(\alpha + \beta + 1)_{n-k}}{(\beta)_{n-k+1}} P_{n-k}^{(\alpha,\beta)}(x) =$$

$$= \frac{\alpha + \beta + 2n}{2n} P_{n-1}^{(\alpha+1,\beta)}(x)$$

it follows

$$R_n^{(\alpha,\beta)}(x, 1) = \frac{x - \omega}{x - 1}\left(P_n^{(\alpha,\beta)}(x) - \frac{\alpha + \beta}{n} P_{n-1}^{(\alpha,\beta)}(x)\right) =$$

$$= \frac{\alpha + \beta + 2n}{2n}(x - \omega)\, P_{n-1}^{(\alpha+1,\beta)}(x)$$

b) Legendre polynomials. For $\alpha = \beta = 0$ we find from (4)

$$R_n^{(0,0)}(x, 1) = \frac{x - \omega}{x - 1}(p_n(x) - p_{n-1}(x)) =$$

$$= (x - \omega)\, P_{n-1}^{(1,0)}(x)$$

where

$$n!P_{n-1}^{(1,0)}(x) = \sum_{n=o}^{n-1} (2n-2k+1)\, P_{n-k}(x)$$

c) Laguerre polynomials. We have

$$a_n = -1/n, \quad b_n = (2n-1+\alpha)/n, \quad c_n = (n-1+\alpha)/n, \quad (a, b) = (0, \infty)$$

Then

$$R_n^{(\alpha)}(x) = n\frac{x-\omega}{x}\left(L_n^{(\alpha)}(x) - \frac{\alpha+n}{n} L_{n-1}^{(\alpha)}(x)\right) =$$
$$= (\omega - x)\, L_{n-1}^{(\alpha+1)}(x).$$

4. *Applications.* Some definite integrals involving orthogonal polynomials may be calculated from the relations given above. We have

$$1^\circ \qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta R_n^{(\alpha,\beta)}(x)\, dx =$$
$$= 2^{\alpha+\beta} \frac{\alpha+\beta+2n}{n} \cdot B(\beta+n, \alpha+1)\,(1-\omega)$$

where $B(p, q)$ is Beta function.

$$2^\circ \qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta R_n^{(\alpha,\beta)}(x) R_m^{(\gamma,\sigma)}\, dx =$$
$$= 2^{\alpha+\beta} \frac{\alpha+\beta+2n}{n} B(\beta+n, \alpha+1)\,(1-\omega)^2 \binom{\gamma+m}{m}$$

$$3^\circ \qquad \int_{-1}^{1} (1-x)^\alpha (1+x)^\beta R_n^{(\alpha,\beta)}(x) L_m^{(\gamma)}(x)\, dx =$$
$$= 2^{\alpha+\beta} B(\beta+n, \alpha+1)\, \frac{(\alpha+\beta+2n)}{2}\, (1-\omega)^2\, \frac{(\gamma+2)}{m!}\, F(-m, \gamma, 1)$$

$$4^\circ \quad \int_{o}^{n} e^{-x} x^\alpha R_n^{(\gamma)}(x) R_n^{(\alpha,\beta)}(x)\, dx = \frac{\omega^2}{n}\binom{\alpha+n}{n}\Gamma(\gamma+1)$$

$$5^\circ \quad \int_{0}^{\infty} e^{-x} x^\alpha R_n^{(\alpha)}(x) R_m^{(\ )}(x)\, dx = \frac{\omega^2}{mn}\binom{\beta+m}{m}\Gamma(\gamma+1).$$

REFERENCES

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ЕДНА ЗАБЕЛЕШКА ЗА ОРТОГОНАЛНИТЕ ПОЛИНОМИ

(Резиме)

При дадена класа ортогонални полиноми дадена е постапка за формулирање нова класа ортогонални полиноми кои имаат една заедничка нула.