A Class of Generating Functions for the Jacobi and Related Polynomials

Studies in Applied Mathematics, Vol. 78, n°3, (1988), 177-182

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This paper aims at presenting a general class of mixed generating functions for the Jacobi polynomials. It is also shown how the main generating function can be suitably applied to yield numerous further results involving Jacobi polynomials and various other polynomials associated with them.

1. Introduction

Put

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \binom{n+\alpha}{n-k}\binom{n+\beta}{k}\left(\frac{x-1}{2}\right)^k\left(\frac{x+1}{2}\right)^{n-k}, \tag{1.1}$$

where $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomial of degree n in x and of order α, β (cf. [5, Chapter 16] and [12, Chapter 4]). With a view to obtaining bilinear, bilateral, or mixed multilateral generating functions for the modified Jacobi polynomials $P_n^{(\alpha-n,\beta-n)}(x)$, several workers (see, for example, [1], [2], [3], [6], [8], and [9]) have successfully applied the generating function

$$\sum_{n=0}^{\infty}\binom{m+n}{n}P_{m+n}^{(\alpha-n,\beta-n)}(x)t^n$$
$$= \left[1+\tfrac{1}{2}(x+1)t\right]^{\alpha}\left[1+\tfrac{1}{2}(x-1)t\right]^{\beta} P_m^{(\alpha,\beta)}\left(x+\tfrac{1}{2}(x^2-1)t\right), \tag{1.2}$$

which, as observed by Singhal and Srivastava [6, p. 759], is the special case $y=1$ of a bilinear generating function due to Srivastava [7, p. 465, Equation (21)].

One of the most general applications of the generating relation (1.2) of the type indicated above yields a mixed multilateral generating function for $P_n^{(\alpha-n,\beta-n)}(x)$, which was given by Srivastava [8, p. 230, Corollary 5] and which has since been reproduced in the latest book on the subject by Srivastava and Manocha [10, p. 423, Corollary 5]. The object of the present note is to develop a substantially more general class of mixed generating functions for the Jacobi polynomials as yet another interesting consequence of (1.2). We also show how our main generating function (2.3) below can be suitably applied to yield numerous further results involving Jacobi polynomials and various other polynomials associated with them.

2. The main result

Our main generating function for the Jacobi polynomials is contained in the following

THEOREM. *Corresponding to a nonvanishing function $\Omega_\mu(y_1,\ldots,y_s)$ of s variables $y_1,\ldots,y_s$ $(s\geq 1)$ and of (complex) order μ, let*

$$\Lambda_{m,p,q}^{(\rho,\sigma)}[x;y_1,\ldots,y_s;t] = \sum_{n=0}^{\infty} a_n P_{m+qn}^{(\alpha-\rho qn,\beta-\sigma qn)}(x)\Omega_{\mu+pn}(y_1,\ldots,y_s)t^n,$$
$$a_n \neq 0, \tag{2.1}$$

where p and q are positive integers, and ρ and σ are suitable complex parameters. Also, for an integer $m \geq 0$, let

$$\Phi_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x; y_1,\dots,y_s; z)$$

$$= \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k P_{m+n}^{(\alpha-n+\rho qk,\beta-n+\sigma qk)}(x)\Omega_{\mu+pk}(y_1,\dots,y_s)z^k. \quad (2.2)$$

Then

$$\sum_{n=0}^{\infty} \Phi_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x; y_1,\dots,y_s; z)t^n$$

$$= [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}$$

$$\times \Lambda_{m,p,q}^{(1-\rho,1-\sigma)}\Big[x+\tfrac{1}{2}(x^2-1)t; y_1,\dots,y_s;$$

$$zt^q\{1+\tfrac{1}{2}(x+1)t\}^{(\rho-1)q}\{1+\tfrac{1}{2}(x-1)t\}^{(\sigma-1)q}\Big], \quad (2.3)$$

provided that each side exists.

Proof: For convenience, let Δ denote the left-hand side of the generating function (2.3). Substituting for the polynomials

$$\Phi_{n,m,p,q}^{\alpha,\beta,\rho,\sigma}(x; y_1,\dots,y_s; z)$$

from (2.2) into the left-hand side of (2.3), we have

$$\Delta = \sum_{n=0}^{\infty} t^n \sum_{k=0}^{[n/q]} \binom{m+n}{n-qk} a_k P_{m+n}^{(\alpha-n+\rho qk,\beta-n+\sigma qk)}(x)\Omega_{\mu+pk}(y_1,\dots,y_s)z^k$$

$$= \sum_{k=0}^{\infty} a_k \Omega_{\mu+pk}(y_1,\dots,y_s)(zt^q)^k$$

$$\times \sum_{n=0}^{\infty} \binom{m+qk+n}{n} P_{m+qk+n}^{(\alpha+(\rho-1)qk-n,\beta+(\sigma-1)qk-n)}(x)t^n,$$

by interchanging the order of the double summation involved.

The inner series can be summed by applying the generating relation (1.2), with m, α, and β replaced by $m+qk$, $\alpha+(\rho-1)qk$, and $\beta+(\sigma-1)qk$, respectively ($k=0,1,2,\dots$), and we thus find that

$$\Delta = [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}$$

$$\times \sum_{k=0}^{\infty} a_k P_{m+qk}^{(\alpha+(\rho-1)qk,\beta+(\sigma-1)qk)}\big(x+\tfrac{1}{2}(x^2-1)t\big)\Omega_{\mu+pk}(y_1,\dots,y_s)$$

$$\times \Big\{zt^q[1+\tfrac{1}{2}(x+1)t]^{(\rho-1)q}[1+\tfrac{1}{2}(x-1)t]^{(\sigma-1)q}\Big\}^k.$$

Interpreting this last infinite series by means of the definition (2.1), we arrive at once at the right-hand side of the assertion (2.3).

This evidently completes the proof of the theorem under the assumption that the double series involved in the first two steps of our proof are absolutely convergent. Thus, in general, the theorem holds true for those values of the various parameters and variables involved for which each side of the assertion (2.3) exists.

3. Applications

At the outset we should remark that, in the special case when $\rho = \sigma = 0$, our main assertion (2.3) reduces immediately to the aforementioned result of Srivastava ([8, p. 230, Corollary 5]; see also [10, p. 423, Corollary 5]). Furthermore, our theorem with $q = \rho = \sigma = 1$ is essentially the same as a recent result of Das [4, p. 99, Theorem VI], who proved this special case of our assertion (2.3) in a markedly different manner by employing certain differential operators.

In order to illustrate how our theorem can be applied to derive various generating functions involving Jacobi polynomials, we recall the familiar result ([10, p. 145, Equation (31)]; see also [11, p. 58, Equation (1.11)] for an alternative form):

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{u}(\gamma_j)_n}{\prod_{j=1}^{v}(\delta_j)_n} P_n^{(\alpha,\beta)}(x)t^n$$

$$= F^{u+2:0;0}_{v:1;1}\left[\begin{matrix}\alpha+1,\beta+1,(\gamma_u): & —; & —; \\ (\delta_v): & \alpha+1; & \beta+1;\end{matrix}\ \tfrac{1}{2}(x-1)t,\ \tfrac{1}{2}(x+1)t\right], \tag{3.1}$$

where $(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda)$; $F^{p:r;u}_{q:s;v}$ denotes a general double hypergeometric series defined by [10, p. 63, Equation (16)]:

$$F^{p:r;u}_{q:s;v}\left[\begin{matrix}(a_p): & (\alpha_r); & (\gamma_u); \\ (b_q): & (\beta_s); & (\delta_v);\end{matrix}\ x, y\right]$$

$$= \sum_{m,n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{m+n}\prod_{j=1}^{r}(\alpha_j)_m\prod_{j=1}^{u}(\gamma_j)_n}{\prod_{j=1}^{q}(b_j)_{m+n}\prod_{j=1}^{s}(\beta_j)_m\prod_{j=1}^{v}(\delta_j)_n}\frac{x^m}{m!}\frac{y^n}{n!}; \tag{3.2}$$

and, for convenience, (a_p) abbreviates the array of p parameters $a_1,\ldots,a_p$, with similar interpretations for (b_q), (α_r), (β_s), etc.

In view of (3.1), we now set

$$a_n = \frac{\prod_{j=1}^{u}(\gamma_j)_n}{\prod_{j=1}^{v}(\delta_j)_n}, \qquad \Omega_\mu(y_1,\ldots,y_s) \equiv 1, \qquad m = 0,$$

and (for simplicity) let $q = \rho = \sigma = 1$. We then find from our theorem that

$$\sum_{n=0}^{\infty} \Psi_n^{(\alpha,\beta)}(x;z)t^n$$

$$= [1+\tfrac{1}{2}(x+1)t]^{\alpha}[1+\tfrac{1}{2}(x-1)t]^{\beta}$$

$$\times F^{u+2:0;0}_{v:1;1}\left[\begin{matrix}\alpha+1,\beta+1,(\gamma_u): & —; & —; \\ (\delta_v): & \alpha+1; & \beta+1;\end{matrix}\ z\xi(x,t),\ z\eta(x,t)\right], \tag{3.3}$$

where

$$\Psi_n^{(\alpha,\beta)}(x;z) = \sum_{k=0}^{n}\binom{n}{k}\frac{\prod_{j=1}^{u}(\gamma_j)_k}{\prod_{j=1}^{v}(\delta_j)_k}P_n^{(\alpha-n+k,\beta-n+k)}(x)z^k \tag{3.4}$$

and

$$\xi(x,t) = \tfrac{1}{2}(x-1)t[1+\tfrac{1}{2}(x+1)t], \qquad \eta(x,t) = \tfrac{1}{2}(x+1)t[1+\tfrac{1}{2}(x-1)t]. \tag{3.5}$$

Some very specialized cases of the generating function (3.3) were considered earlier by Das [4, pp. 102–104].

For such choices of the coefficients a_n as illustrated above, if the multivariable function

$$\Omega_\mu(y_1,\dots,y_s), \qquad s>1,$$

is expressed as a suitable product of several simpler functions, our theorem would yield various mixed multilateral generating relations for the Jacobi polynomials. Also, since [12, p. 64, Equation (4.22.1)]

$$P_n^{(\alpha,\beta-n)}(x) = \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-1-n,\beta-n)}\left(\frac{x+3}{x-1}\right), \tag{3.6}$$

and since [9, p. 59, Equation (4.1.3)]

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x), \tag{3.7}$$

our theorem can easily be restated in terms of the modified Jacobi polynomials

$$P_n^{(\alpha-n,\beta)}(x) \quad \text{and} \quad P_n^{(\alpha,\beta-n)}(x),$$

as was done (in the special case $\rho=\sigma=0$) by Srivastava ([8, p. 230, Corollaries 6 and 7]; see also [10, p. 423, Corollary 6; p. 424, Corollary 7]).

Finally, since [12, p. 103, Equation (5.3.4)]

$$L_n^{(\alpha)}(x) = \lim_{\beta\to\infty} P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right), \tag{3.8}$$

analogous results for the Laguerre polynomials

$$L_n^{(\alpha)}(x) \quad \text{and} \quad L_n^{(\alpha-n)}(x),$$

or for various polynomials associated with them, can be deduced as appropriate limiting cases of our theorem or of its variations using (3.6) and (3.7). We omit the details involved in deriving, in this manner, the indicated generalizations of Corollaries 8, 9, 10, and 11 of Srivastava ([8, pp. 231–233]; see also [10, pp. 424–426]).

References

1. A. K. Agarwal and H. L. Manocha, On further extension of bilateral generating functions for certain special functions, *Simon Stevin* 54:179–191 (1980).
2. A. K. Agarwal and H. L. Manocha, A class of trilateral generating functions for certain special functions, *Bull. Calcutta Math. Soc.* 76:79–82 (1984).
3. S. K. Chatterjea, An extension of a class of bilateral generating functions for certain special functions, *Bull. Inst. Math. Acad. Sinica* 5:323–331 (1977).
4. S. Das, On partial differential operators for Jacobi polynomials, *Pure Math. Manuscript* 3:95–104 (1984).
5. E. D. Rainville, *Special Functions*, Macmillan, New York, 1960; reprinted by Chelsea, Bronx, N.Y., 1971.
6. J. P. Singhal and H. M. Srivastava, A class of bilateral generating functions for certain classical polynomials, *Pacific J. Math.* 42:755–762 (1972).
7. H. M. Srivastava, Some bilinear generating functions, *Proc. Nat. Acad. Sci. U.S.A.* 64:462–465 (1969).
8. H. M. Srivastava, Some bilateral generating functions for a certain class of special functions. I, II, *Nederl. Akad. Wetensch. Indag. Math.* 42:221–233, 234–246 (1980).
9. H. M. Srivastava and J.-L. Lavoie, A certain method of obtaining bilateral generating functions, *Nederl. Akad. Wetensch. Indag. Math.* 37:304–320 (1975).
10. H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted (Ellis Horwood, Chichester), Wiley, New York, 1984.
11. R. Srivastava, Some generating functions and hypergeometric transformations, *Bull. Korean Math. Soc.* 22:57–61 (1985).
12. G. Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloq. Publ., Vol. 23, 4th ed., Amer. Math. Soc., Providence, R.I., 1975.