NOTE ON "A TANTALIZING PROBLEM" CONCERNING THE LUCAS NUMBERS

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Abstract. An alternative method for the Sum (1) is given.

1. Recently, R.P.Backstrom (1) for the sum of the series

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} \tag{1}$$

by two methods gives different results, namely

1° $\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} = .64452\ 17830\ 67274\ 44209\ 92731\ 19038$

and

2° $\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} = .64452\ 17303\ 08756\ 88440\ 03306\ 51529$

L_n is the Lucas number.

In the same paper, the author invites the readers to investigate these results and to determine the true value of this series.

The purpose of this Note is, by another method, developing the series (1), to give a new solution of this problem.

2. It is known that

$$L_{2n} + 2 = \alpha^{2n} + \beta^{2n} + 2$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \text{ and } \beta = \frac{1-\sqrt{5}}{2} \tag{2}$$

Then

$$L_{2n} + 2 = (\alpha^n + \frac{1}{\alpha^n})^2, \text{ since } \alpha\beta = -1$$

and we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} = \sum_{n=0}^{\infty} \frac{1}{(\alpha^n + \frac{1}{\alpha^n})^2} \tag{3}$$

We note that

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} = \sum_{n=0}^{\infty} \frac{1}{(\alpha^n + \frac{1}{\alpha^n})^2} = \sum_{n=0}^{\infty} \left| \frac{1}{(\alpha^{2n} + \frac{1}{\alpha^{2n}})^2} = \frac{1}{(\alpha^{2n+1} + \frac{1}{\alpha^{2n+1}})^2} \right|$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{L_{2n}^2} + \frac{1}{5F_{2n+1}^2}\right)$$

where $\sqrt{5}F_n = \alpha^n - \beta^n$.

3. Using the series |2|

$$\sum_{n=1}^{\infty} \frac{z^{2^{n-1}}}{1-z^{2^n}} = \frac{z}{1-z}$$

with $z = \frac{\beta}{\alpha}$, we obtain

$$\beta = -\frac{1}{F_2} + \frac{1}{F_4} + \frac{1}{F_8} + \ldots + \frac{1}{F_{2^n}} + \ldots +$$

$$= -\frac{1}{L_1} + \frac{1}{L_1 L_2} + \frac{1}{L_1 L_2 L_4} + \ldots + \frac{1}{L_1 L_2 L_4 \ldots L_{2^{n-1}}} + \ldots,$$

or by (2) we have

$$\frac{\sqrt{5}-1}{2} = 1-(\frac{1}{3} + \frac{1}{3\cdot 7} + \frac{1}{3\cdot 7\cdot 47} + \frac{1}{3\cdot 7\cdot 47\cdot 2207} +$$

$$+ \frac{1}{3\cdot 7\cdot 47\cdot 2207\cdot 4870847} +$$

$$+ \frac{1}{3\cdot 7\cdot 47\cdot 2207\cdot 4870847\cdot 23725150497407} + \ldots),$$

and

$$\frac{\sqrt{5}+1}{1} = 1 + \frac{\sqrt{5}-1}{2}.$$

Substituting these values into (3) we have

$$\sum_{n=0}^{\infty} \frac{1}{L_{2n}+2} = .64452\ 17830\ 67182\ 34\ 432\ 15783\ 48807\ 649$$

R E F E R E N C E S

[1] Backstrom, R.P.: On Reciprocal Series Relate to Fibonacci Numbers with Subscripts in Arithmetic Progression the Fibonacci Quarterly, 19 (1981), 14-21

[2] Polya, G., Szego, G.: Aufgaben und Lehrsatzen aus der Analysis, zweiter Band, Springer-Verlag, Berlin, 1925

ЕДНА ЗАБЕЛЕШКА ЗА СУМИ НА LUCAS-ОВИ БРОЕВИ

Р е з и м е

Се дава една алтернативна метода за сумата (1), која се однесува за броевите на Lucas.