ON CERTAIN DOUBLE SERIES INVOLVING GENERALIZED HYPERGEOMETRIC SERIES

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§1. The object of this paper is to evaluate certain double series involving G-function of two variables (cf. Agarwal [1]) in terms of another G-function of two variables. Certain similar results for Meijer's G-function and MacRobert's E-function have also been deduced as special cases of the results discussed herein.

The following result due Jain [4] shall be used:

$$\sum_{r=0}^{m}\sum_{s=0}^{n}\frac{[-m]_r\,[-n]_s\,[c-a]_r\,[a]_s\,[c-b]_r\,[b]_s}{[1]_r\,[1]_s\,[c]_{r+s}\,[1-a-b+c-m]_r\,[1+a+b-c-n]_s}=\tag{1.1}$$
$$=\frac{[a]_m\,[c-a]_n\,[b]_m\,[c-b]_n}{[c]_{m+n}\,[a+b-c]_m\,]c-a-b]_n}.$$

§2. Now, we shall establish the following result:

$$\sum_{u=0}^{m}\sum_{v=0}^{n}\frac{[-m]_u\,[-n]_v}{[1]_u\,[1]_v\,[1-a-b+c-m]_u\,[1+a+b-c-n]_v}\times\tag{2.1}$$
$$\times\, G^{r,\,2+\nu_1,\,2+\nu_2,\,m_1,\,m_2}_{p,\,[2+t:2+t'],\,s+1,\,[q:q']}\left[\begin{array}{c|c} x & (\varepsilon_p) \\ & c-a+u,\ b+v,\ (\gamma_t);\ c-b+u,\ a+v,\ (\gamma'_{t'}) \\ y & (\delta_s),\ c+u+v \\ & (\beta_q);\ (\beta'_{q'}) \end{array}\right]=$$
$$=\frac{1}{[a+b-c]_m\,[c-a-b]_n}\times$$
$$\times\, G^{r,\,2+\nu_1,\,2+\nu_2,\,m_1,\,m_2}_{p,\,[t+2:t'+2],\,s+1,\,[q:q']}\left[\begin{array}{c|c} x & (\varepsilon_p) \\ & c-a+n,\ b+m,\ (\gamma_t);\ c-b+n,\ a+m,(\gamma'_{t'}) \\ y & (\delta_s),\ c+m+n \\ & (\beta_q);\ (\beta'_{q'}) \end{array}\right],$$

where G-function appearing in (2.1) is G-function of two variables. (It is not necessary to take the same number of parameters γ and γ' and so also β and β' as taken by Agarwal [1].).

(2.1) Holds under the following set of conditions:

$$0 \leqslant r \leqslant p,\ 0 \leqslant \nu_1 \leqslant t,\ 0 \leqslant m_1 \leqslant q,\ 0 \leqslant \nu_2 \leqslant t',\ 0 \leqslant m_2 \leqslant q',$$

$$p + q + s + t < 2(r + \nu_1 + m_1),$$

$$p + q' + s + t' < 2(r + \nu_2 + m_2),$$

$$|\arg x| < \pi[r + \nu_1 + m_1 - (p + q + s + t)/2],$$

$$|\arg y| < \pi[r + \nu_2 + m_2 - (p + q' + s + t')/2].$$

To prove (2.1), we replace the G-function of two variables on the left of it by its equivalent double integral (zf. Agarwal [1]), interchange the order of summation and integration and sum the inner double series with the help of known result (1.1). The double integral is now replaceable by its equivalent G-function of two variables. This completes the proof of (2.1).

If we take $m_2 = 1,\ t = t'\ q = q',\ q \geqslant t,\ p = 0 = s = y = \beta_1'$ in (2.1), we get, after some simplification,

$$\sum_{n=0}^{m} \sum_{\nu=0}^{n} \frac{[-m]_u\,[-n]_v\,[c-b]_u\,[a]_v}{[1]_u\,[1]_\nu\,[1-a-b+c-m]_u\,[1+a+b-c-n]_v} \times$$

$$\times\, G^{m_1,\,2+\nu_1}_{t+2.\,q+1}\left[x\Big|{}^{1-c+a-u,\ 1-b-\nu,\ 1-(\gamma_t)}_{(\beta_q),\ 1-c-u-\nu}\right] = \tag{2.2}$$

$$= \frac{[c-b]_n\,[a]_m}{[a+b-c]_m\ [c-a-b]_n}\, G^{m_1,\,2+\nu_1}_{t+2,\,q+1}\left[x\Big|{}^{1-c+a-n,\ 1-b-m,\ 1-(\gamma_t)}_{(\beta_q),\ 1-c-m-n}\right]$$

where

$$0 \leqslant m_1 \leqslant q,\ 0 \leqslant \nu_1 \leqslant t,$$

$$t + q < 2(m_1 + \nu_1),$$

$$|\arg x| < \pi[m_1 + \nu_1 - (t + q)/2].$$

The conditions $t = t',\ q = q',\ q \geqslant t$ can now be waived off by analitic continuation.

(2.2) gives the corresponding result for Meijer's G-function.

Next, if we take $m_1 = 1,\ \beta_1 = 0,\ \nu_1 = t$, replace q by $q + 1$ and put $\beta_{r+1} = \beta_r (r = 1, 2, \ldots, q)$ and then replace the $G(x)$-function by its equivalent $G(1/x)$-function, we get, after some simplification,

$$\sum_{u=0}^{m} \sum_{v=0}^{n} \frac{[-m]_u [-n]_v [c-b]_u [a]_v}{[1]_u [1]_v [1-a-b+c-m]_u [1+a+b-c-n]_v} \times$$

(2.3) $$\times E[2+t;\ c-a+u, b+v, (\gamma_t): q+1; 1-(\beta_q),\ c+u+v: 1/x] =$$

$$= \frac{[c-b]_n [a]_m}{[a+b-c]_m [c-a-b]_n} \times$$

$$\times E[2+t; c-a+n, b+m, (\gamma_t): q+1; 1-(\beta_q),\ c+m+n: 1/x],$$

where $|\arg 1/x| < (t-q)\pi/2$.

(2.3) Gives the corresponding result for MacRobert's *E*-function.

It may be remarked here that a number of such results, using the known summations (Carlitz [*2*]),, can also be established under appropriate convergence conditions.

REFERENCES

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