SOME INTEGRALS INVOLVING THE H-FUNCTION AND GENERALIZED LEGENDRE FUNCTIONS

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1. INTRODUCTION:

The object of the present paper is to evaluate some integrals involving Fox's H-function and generalized Legendre functions introduced by Kuipers and Meulenbeld. The generalized Legendre functions reduce to associated Legendre functions on setting $m = n$ and to Legendre functions on setting $m = n = o$. Also, H-function is a very general function. Thus, on specializing the parameters of these functions in the integrals we can get many new as well as known results as particular cases.

The functions $P^{m,n}_{k}(z)$ and $Q^{m,n}_{k}(z)$, solutions of the differential equation:

$$(1.\ 1)\quad (1\text{-}z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + \left\{k(k+1) - \frac{m^2}{2(1-z)} - \frac{n^2}{2(1+z)}\right\} w = 0,$$

introduced in [4] by Kuipers-Meulenbeld, have been defined for all points of the z-plane in which a crosscut exists along the real axis from 1 to $-\infty$ and in [6] for the real values of z on the cross-cut for $-1 < z < 1$. These functions have been called generalized Legendre functions.

The H-function has been introduced by Fox [3, *p.* 408] and its conditions of validity, asymptotic expansions and analytic continuations have been discussed by Braaksma [1]. Following the definition given by Braaksma [1, *pp.* 239-241], it will be represented as follows:

$$(1.\ 2)\quad H^{l,\,u}_{r,\,s}\left[z \,\middle|\, \begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right] = \frac{1}{2\pi i}\int_T \frac{\prod_{j=1}^{l}\Gamma\,(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma\,(1-a_j+\alpha_j\xi)}{\prod_{j=l+1}^{s}\Gamma\,(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma\,(a_j-\alpha_j\xi)}\, z^{\xi}\, d\xi,$$

where $\{(a_r, \alpha_r)\}$, stands for the set of parameters $(a_1\,\alpha_1)\,, \cdots\, (a_r, \alpha_r)\cdot$

In what follows, for the sake of brevity

$$\sum_1^s (\beta_j) - \sum_1^r\ (\alpha_j) \equiv A \text{ and } \sum_1^l (\beta_j) - \sum_{l+1}^s (\beta_j) + \sum_1^u (\alpha_j) - \sum_{u+1}^r\ (\alpha_j) \equiv B.$$

2. The integrals to be established are as under:

$$(2.\ 1)\quad \int_1^\infty (x-1)^{\,p}\,(x+1)^q\ \ Q^{m,\,n}_k(z)\ H^{l,\,u}_{r,\,s}\left[z\,(x-1)^\lambda\,(x+1)^\delta \,\middle|\, \begin{matrix}\{(a_j,\ \alpha_j)\}\\ \{(b_s,\ \beta_s)\}\end{matrix}\right] dx$$

$$= 2^{\,p+q-\frac{m}{2}+\frac{n}{2}}\, e^{\,\pi\, im}\quad \Gamma\,(1+\mathrm{k}+\frac{1}{2}(m+n))\,\Gamma\,(1+\mathrm{k}\,\frac{1}{2}\,(m-n))\ \times$$

$$\times\sum_{N=o}^{\infty}\frac{\left(1+\mathrm{k}-\frac{1}{2}(m-n)\right)_N\ \left(1+\mathrm{k}-\frac{1}{2}\ (m+n)\right)_N}{N!\,\Gamma(2\,\mathrm{k}+2+N)}\times$$

$$\times\ H^{l+1,\,u+1}_{r+2,s+1}\left[2^{\,\delta+\lambda}\,z \,\middle|\, \begin{matrix}\left(\frac{m}{2}-p,p,\lambda\right),\ \{\,(a_r,\ \alpha_r)\},\ (1+\mathrm{k}-\mathrm{q}-\frac{m}{2},\ \delta)\\ (\mathrm{k}-\mathrm{p}-\mathrm{q}+N,\ \lambda+\delta)\,,\ \{\,(\mathrm{b}_s,\ \beta_s)\}\end{matrix}\right],$$

where $A \geqslant 0$, $B > 0$, $|\ \arg\ \mathrm{z}\ | < \frac{1}{2}B\pi$, $Re\ (p+\lambda b_j/\beta_j\,\lambda) > \frac{1}{2}\,|\,Re m\,|-1$ $(j=1,\,2,\,\ldots,\,l)$, $Re\ (p+q-k+(\lambda+\delta)\,(a_i-1)\,\alpha_i) < 0\ (j=1{,}2,\ldots,\ \mathrm{u})$, and $\delta > 0, \lambda \geqslant 0$ (*or* $\lambda > 0,\ \delta \geqslant 0$).

$$(2.\ 2)\ \int_1^\infty (x-1^p\ (x+1)^q\ \ Q^{m,\,n}_k\,(x)\ H^{l,\,u}_{r,\,s}\left[\ z\left(\frac{x-1}{x+1}\right)^{\delta} \,\middle|\, \begin{matrix}\{(a_r,\ \alpha_r)\}\\ \{(b_s\ \beta_s)\}\end{matrix}\right] dx$$

$$= 2^{\,p+q-\frac{m}{2}+\frac{n}{2}}\, e^{\,\pi im}\quad \Gamma(k+1+\frac{1}{2}\ (m+n\,))\ \Gamma\,(1+k+\frac{1}{2}\,(m-n))\ \times$$

$$\times \sum_{N=0}^{\infty} \frac{\Gamma(k-p-q+N)\ (1+k-\frac{1}{2}(m-n))_N\,(1+k-\frac{1}{2}(m+n))_N}{N!\ \Gamma(2k+2+N)}$$

$$\times H^{l,u+1}_{r+1,s+1}\left[\ z\ \middle|\ \begin{matrix}(\frac{m}{2}-p,\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(q-k-N+\frac{m}{2},\delta)\end{matrix}\right],$$

$$\text{where } \delta>0, A\geqslant 0,\ B>0, |\arg z|<\frac{1}{2}B\pi,$$

$$Re\,(p+q-k)>0,\ Re p+\delta\, b_j/\beta_j)>\frac{1}{2}\,|Re m|-1\ (j=1,2,\ldots,l).$$

$$(2.\ 3)\int_1^{\infty}(x-1)^{-\frac{1}{2}m}(1+x)^{-p}P^{m,n}_k(x)\,H^{l,u}_{r,s}\left[z(1+x)^{-\delta}\middle|\begin{matrix}\{(a_r,\varkappa_r)\}\\ \{(b_s,\beta_s)\}\end{matrix}\right]dx$$

$$=2^{1-m-p+\frac{n}{2}}H^{l,u+2}_{r+2,s+2}\left[2^{-\delta}z\ \middle|\ \begin{matrix}(1\text{-}p\text{-}k\text{-}\frac{m}{2},\delta),(2\text{-}p+k\text{-}\frac{m}{2},\delta),\{(a_r,\alpha_r)\}\\ \{(b_s,\beta_s)\},(1\text{-}p+\frac{n}{2},\delta),(1\text{-}p\text{-}\frac{n}{2},\delta)\end{matrix}\right],$$

$$\text{provided } \delta>0,\ A\geqslant 0,\ B>0, |\arg z|<\frac{1}{2}B\pi,$$

$$Re\ (m)<1,\ Re\,(p+k+\frac{m}{2}+\delta\,(a_i-1)/\alpha_i>0,$$

$$Re\,(p-k+\frac{m}{2}-1+\delta)\,a_i-1)/\alpha_i)>0,\ (i=1,2,\ldots,u).$$

PROOF: To estabilsh the integral (2. 1), expressing the H — function in the integrand as Mellin — Barnes type of integral (1. 2), interchanging the order of integration which is justifiable due to the absolnte convergence of the integrals involved in the process, we get

$$(2.\ 4)\ \frac{1}{2\pi i}\int_T \frac{\prod_{j+1}^{l}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{u}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=l+1}^{s}\Gamma(1-b_j+\beta_j\xi)\prod_{j=u+1}^{r}\Gamma(a_j-\varkappa_j\xi)}\,z^{\xi}\times$$

$$\times \int_1^\infty (x-1)^{p+\lambda\xi} (x+1)^{q+\delta\xi} Q_k^{m,n}(x)\, dx\, d\xi .$$

Evaluating the inner integral with the help of [7, *p*. 289 [(23)], i e.

$$(2.\ 5.) \int_1^\infty (x-1)^p (x+1)^q e^{-\pi i m} Q_k^{m,n}(x)\, dx$$

$$= 2^{p+q-\frac{1}{2}m+\frac{1}{2}n} \cdot \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)\Gamma(p-\frac{1}{2}m+1)(\Gamma(-p-q+k)}{\Gamma(2k+2)\Gamma(k-q-\frac{1}{2}m+1)} \times$$

$$\times {}_3F_2(\beta+1, \delta+1, k-p-q;\ 2k+2, k-q-\frac{1}{2}m+1; 1),$$

where $\frac{1}{2}|Re m|-1 < Re(p) < Re(k-q), \alpha=k+\frac{1}{2}(m+n), \beta=k-\frac{1}{2}(m-n),$

$\gamma=k+\frac{1}{2}(m-n)$ and $\delta=k-\frac{1}{2}(m+n)$; expressing the hypergeometric function as series and changing the order of summation and integration in view of [2, *p*. 176 (75)], which is permissible uuder the conditions given in (2. 1) and (2. 5); and again applying (1. 2), the definition of the H — function, the value of the integral is obtained.

Similarly (2. 2) can be evaluated by using (2. 5) and (2. 3) can be established with the help of [7, *p*. 208 (15)] viz.,

$$(2.\ 6) \int_1^\infty (x-1)^{-\frac{1}{2}m} (x+1)^{-p} P_k^{m,n}(x)\, dx$$

$$= 2^{1-m+\frac{1}{2}n-p} \cdot \frac{\Gamma(p+k+\frac{1}{2}m)\Gamma(p-k+\frac{1}{2}m-1)}{\Gamma(p-\frac{1}{2}n)\Gamma(p+\frac{1}{2}n)},$$

where $Re.(m) < 1$, $Re\,(p+k+\frac{1}{2}m) > 0$ and $Re\,(p-k+\frac{1}{2}m-1) > 0$.

3. In this section, we discuss on interesting particular case of the result (2. 2).

In (2. 2), setting $q = -p-k$ and on the righthand side expressing the H — function as Mellin — Barnes type integral, interchanging the order of integration and summation, evaluating the series inside the integral with the help of Gauss' theorem [5, p 144] and again using (1. 2), the definition of the H — function, we obtain

$$\text{(3 1)} \quad \int_1^{\infty} (x-1)^{p} (x+1)^{-p-k-2} Q_k^{m,n}(x) H_{s,r}^{l,u} \left[z \left(\frac{x-1}{x+1}\right)^{\delta} \middle| \begin{matrix} \{(a_r, \alpha_r)\} \\ (\{b_s, \beta_s)\} \end{matrix} \right] dx$$

$$= 2^{-k-\frac{m}{2}+\frac{n}{2}-2} \quad \Gamma(1+k+\frac{1}{2}(m+n)) \, \Gamma(1+k+\frac{1}{2}(m-n)) \times$$

$$\times H_{r+2,s+2}^{l,u+2} \left[z \middle| \begin{matrix} \left(\frac{m}{2}-p, \delta\right), (-p-\frac{m}{2}, \delta), \{(a_r, \alpha_r)\} \\ \{(b_s, \beta_s)\}, (-1+\frac{n}{2}-p-k, \delta), (-1-p-k-\frac{n}{2}, \delta) \end{matrix} \right],$$

provided $\delta > 0$, $A \geqslant 0$, $B > 0$; $|\arg z| < \frac{1}{2} B\pi$,

$Re\ (k) > -1$, $Re\,(p + \delta b_j / \beta_j) > \frac{1}{2} |\, Re m | - 1 |$ $(j = 1, 2, \ldots, l)$.

REFERENCES

1 BRAAKSMA, B, L. J., : „Asymptotic expansions and analytic continuations for a class of Barnes-integrals", Compos. Math. 15 (1963), 239-341.

2. CARSLAW, H. S., : Introduction to the theory of Fourier's series and integrals, Dover Publications, Inc., New York (1950).

3. FOX, C., : „The G and H-functions as symmetrical Fourier kernels", Trans. Amer. Math. Soc., 3, Vol. 98 (1961), 395-429.

4. KUIPERS, L., & MEULENBEŁD, B., : „On a generalization of Legendre associated differential equation I and II", Nederl, Akad. Wetensch. Proc. Ser A. 60 (1957), 336-450,

5, MACROBERT, T. M,, : Functions of a Complex Variable Macmillan & Co., New York (1962).

6. MEULENBELD, B., : „Generalized Legendre' s associated functions for real values of the argument numerically less than unity", Hederl. Akad. Wetensch. Proc. Ser. A, (1958, 555-563.

7. MEULENBELD, B.. & DE SNOO, H. S. V., : „Integrals involving generalized Legendre Functions", Jour of Engineering Mathematics, Vol 1. 1967), 285-291.