SOME ACCELERATIONS OF THE CONVERGENCE OF CERTAIN CLASS OF SEQUENCES

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Abstract. The sequence $\{U_{n+1}/U_n\}$ of the ratios of consecutive numbers U_n, $n=0,1,2,\ldots$ defined by $aU_{n+1}+bU_n+cU_{n-1} = 0$ with $U_0=0$, $U_1=1$ converges to the root λ_1 of $f(x)=ax^2+bx+c=0$, supposing $|\lambda_1| > |\lambda_2|$. Newton's method for the equation $f(x)=0$ with initial approximation 1 produces the subsequence $\{U_{2^n+1}/U_{2^n}\}$. The Halley's iteration method for this equation produces the subsequence $\{U_{3^n+1}/U_{3^n}\}$. Applying the Newton's modified method and the Schröder's iteration method we obtain similar subsequences.

1.1. Let $\{W_n\}$ be a sequence defined by the second-order linear difference equation

$$aW_{n+1} + bW_n + cW_{n-1} = 0 \tag{1}$$

with the initial values W_0 and W_1. To get a simpler relationship for this sequence, we write the last equation (1) in the form [1]

$$aW_{n+1}-\lambda aW_n+(b-\lambda a)(W_n-\lambda W_{n-1}) + (c+b\lambda+a\lambda^2)W_{n-1} = 0.$$

If λ_1,λ_2 are the roots of $a\lambda^2+b\lambda+c=0$, then $\lambda_1+\lambda_2=-b/a$, $\lambda_1\lambda_2=c/a$ and we have

$$W_{n+1} - \lambda_1 W_n = \lambda_2(W_n-\lambda_1 W_{n-1})$$

$$W_{n+1} - \lambda_2 W_n = \lambda_1(W_n-\lambda_2 W_{n-1}),$$

and hence

$$W_{n+1} - \lambda_1 W_n = \lambda_2^n(W_1-\lambda_1 W_0)$$

$$W_{n+1} - \lambda_2 W_n = \lambda_1^n(W_1-\lambda_2 W_0).$$

Substracting we find

$$(\lambda_2-\lambda_1)W_n = (W_1-\lambda_1 W_0)\lambda_2^n - (W_1-\lambda_2 W_0)\lambda_1^n.$$

Therefore, if $\lambda_1\neq\lambda_2$ we have

$$W_n = \frac{(W_1-\lambda_1 W_0)\lambda_2^n-(W_1-\lambda_2 W_0)\lambda_1^n}{\lambda_2-\lambda_1}, \quad \lambda_2\neq\lambda_1.$$

1.2. In the special case when $W_0=0$, $W_1=1$ we obtain the sequence $\{U_n\}$ defined by

$$U_n = \frac{\lambda_2^n-\lambda_1^n}{\lambda_2-\lambda_1}, \quad n=0,1,2,\ldots \tag{2}$$

and if $W_0=2$, $W_1=1$ we have

$$V_n = \lambda_1^n + \lambda_2^n, \quad n=0,1,2,\ldots \tag{3}$$

From (2) and (3) it is easy to verify the identities

$$aU_{m+1}U_{n+1} - cU_mU_n = aU_{m+n+1}, \quad m,n \geq 0 \tag{4}$$

$$aU_{n+1} - cU_{n-1} = aV_n, \quad n \geq 0 \tag{5}$$

2.1. It is known that the sequence $\{U_{n+1}/U_n\}$ of ratios of consecutive numbers U_n, $n=0,1,2,\ldots$ converges linearly to λ_1, supposing $|\lambda_1| > |\lambda_2|$. That is, the number of digits of U_{n+1}/U_n which agree with λ_1 is approximatly a linear function of n. In fact there are constants $\alpha,\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^n < |U_{n+1}/U_n-\lambda_1| < \varepsilon\beta^n$.

J. Gill and G. Miller [2] consider sequences of numbers converging rapidly to λ_1. By Newton's method

$$N(x_n) = x_n - f(x_n)/f'(x_n)$$

for approximating solution of the equation $f(x)=ax^2+bx+c = 0$, they obtain

$$N(U_{n+1}/U_n) = U_{2n+1}/U_{2n}$$

The sequence $\{x_n\}$ generated by Newton's method with $x_o=1$ is defined by $x_n=U_{2^n+1}/U_{2^n}$. The convergence of $\{x_n\}$ is quadratic since there are constants $\alpha,\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^{2n} <| x_n-\lambda_1|<\alpha\varepsilon^{2n}$.

2.2. In the present paper by using procedure of numerical analysis for approximating solutions of the equation $f(x)=0$ we obtain sequences converging more rapidly to λ_1.

Indeed, by Halley's iteration method [3], [5]

$$H(x_n) = x_n-f(x_n)f'(x_n)/(f'^2(x_n)-0,5f(x_n)f''(x_n))$$

for solving the equation $f(x)=0$ with ratios U_{n+1}/U_n as the initial approximation, we find $H(U_{n+1}/U_n)=U_{3n+1}/U_{3n}$. The sequence $\{x^*_n\}$ generated by this method is given by $x^*_n=U_{3^n+1}/U_{3^n}$, $n=0,1,2,\ldots$ which is cubically convergent to λ_1; that is, the number of digits of U_{n+1}/U_n which agree with λ_1 is approximately a cubic function of n. In this case there are constants $\alpha,\beta > 0$ and $\varepsilon < 1$ such that $\alpha\varepsilon^{3n} < |x^*_n-\lambda_1| < \beta\varepsilon^{3n}$.

Next we use the Newton's modified formula, obtained from Newton's method by replacing $f(x)$ by $f(x)/f'(x)$,

$$\overline{N}(x_n) = x_n-f(x_n)f'(x_n)/(f'^2(x_n)-f(x_n)f''(x_n)),$$

from where we have the identity

$$\overline{N}(U_{n+1}/U_n) = V_{2n+1}/V_{2n}$$

The sequence $\{\overline{x}_n\}$ generated by this method is defined by $\overline{x}_n = V_{2^n+1}/V_{2^n}$, $n=0,1,2,\ldots$.

We can similarly apply the Schröder's iteration method [4]

$$S(x_n) = x_n-f(x_n(f'^2(x_n)-f(x_n)f''(x_n))/$$
$$/(f'^3(x_n)-1,5f(x_n)f'(x_n)f''(x_n)+0,5f^2(x_n)f'''(x_n))$$

for solving the equation $f(x)=0$, with ratios U_{n+1}/U_n as initial approximation. We find $S(U_{n+1}/U_n) = V_{3n+1}/V_{3n}$ and $S(V_{n+1}/V_n) = U_{3n+1}/U_{3n}$ from which we obtain the analogous sequence $\{\bar{x}^*_n\}$ with $\bar{x}^* = V_{3^n+1}/V_{3^n}$ which is cubically convergent to λ_1.

3.1. Newton's method for the equation $f(x)=0$ gives

$$N(x_n) = \frac{ax_n^2-c}{2ax_n+b},$$

or if we take the ratios U_{n+1}/U_n as an approximation to λ_1, we have

$$N(U_{n+1}/U_n) = \frac{aU_{n+1}^2-cU_n^2}{U_n(2aU_{n+1}+bU_n)}$$

By the identity (4) we obtain

$$N(U_{n+1}/U_n) = \frac{U_{2n+1}}{U_{2n}}$$

The sequence $\{x_n\}$ generated by Newton's method with $x_0=1$ is defined by $x_n = U_{2^n+1}/U_{2^n}$.

3.2. For the equation $f(x)=0$, the Halley's iteration method gives

$$H(x_n) = \frac{a^2x_n^3-3acx_n-bc}{3a^2x_n^3+3abx_n+(b^2-ac)}.$$

Then

$$H(U_{n+1}/U_n) = \frac{a^2U_{n+1}^3-3acU_{n+1}U_n-bcU_n^3}{U_n(3a^2U_{n+1}^2+3abU_{n+1}U_n+(b^2-ac)U_n^2)}.$$

But by (4) we have

$$\begin{aligned} a^2U_{n+1}^3-3acU_{n+1}U_n-bcU_n^3 &= aU_{n+1}(aU_{n+1}^2-cU_n^2)-cU_n^2(2aU_{n+1}+bU_n) = \\ &= a^2U_{n+1}U_{2n+1}-acU_nU_{2n} = \\ &= a^2U_{3n+1} \end{aligned}$$

and

$$\begin{aligned} &U_n(3a^2U_{n+1}^2+3abU_{n+1}U_n+(b^2-ac)U_n^2) = \\ &= aU_n(aU_{n+1}^2-cU_n^2)+U_n(aU_{n+1}+bU_n)(2aU_{n+1}+bU_n) = \\ &= a^2U_nU_{2n+1}-acU_{n-1}U_{2n} = \\ &= a^2U_{3n} \end{aligned}$$

so that

$$H(U_{n+1}/U_n) = U_{3n+1}/U_{3n}.$$

3.3. The Newton's modified method for $f(x)=0$ gives

$$\bar{N}(x_n) = -\frac{abx_n^2+4acx_n+bc}{2a^2x_n^2+2abx_n+(b^2-2ac)},$$

from where it is

$$\bar{N}(U_{n+1}/U_n) = -\frac{abU_{n+1}^2+4acU_{n+1}U_n+bcU_n^2}{2a^2U_{n+1}^2+2abU_{n+1}U_n+(b^2-2ac)U_n^2}.$$

Using the identities (4) and (5) we obtain

$$abU_{n+1}^2+4acU_{n+1}U_n+bcU_n^2 =$$
$$= aU_{n+1}(bU_{n+1}+2cU_n)+cU_n(2aU_{n+1}+bU_n) =$$
$$= -a(aU_{n+2}U_{n+1}-cU_{n+1}U_n)+c(aU_{n+1}U_n-cU_nU_{n+1}) =$$
$$= -a(aU_{2n+2}-cU_{2n}) = -a^2V_{2n+1}$$

and

$$2a^2U_{n+1}^2+2abU_{n+1}U_n+(b^2-ac)U_n^2 =$$
$$= a^2U_{n+1}^2+(aU_{n+1}+bU_n)^2-2acU_n^2 =$$
$$= a(aU_{n+1}^2-cU_n^2)-c(aU_n^2-cU_{n-1}^2) =$$
$$= a(aU_{2n+1}-cU_{2n-1}) = a^2V_{2n}.$$

Thus we obtain $\bar{N}(U_{n+1}/U_n)=V_{2n+1}/V_{2n}$, from where $\bar{x}_n=V_{2^n+1}/V_{2^n}$. Similarly, $\bar{N}(V_{n+1}/V_n)=U_{2n+1}/U_{2n}$.

<u>3.4</u>. The next cubically convergent sequence is obtained by the Schröder's iteration formula for $f(x)=0$ which gives

$$S(x_n) = -\frac{a^2bx_n^3+6a^2cx_n^2+3abcx_n+c(b^2-2ac)}{2a^3x_n^3+3abx_n^2+3a(b^2-2ac)x_n+b(b^2-3ac)}$$

from where

$$S(U_{n+1}/U_n) = -\frac{a^2bU_{n+1}^3+6a^2cU_{n+1}^2U_n+3abcU_{n+1}U_n^2+c(b^2-2ac)U_n^3}{2a^3U_{n+1}^3+3a^2bU_{n+1}^2U_n+3a(b^2-2ac)U_{n+1}U_n^2+b(b^2-3ac)U_n^3}$$

By (4) and (5) we have

$$a^2bU_{n+1}^3+6a^2cU_{n+1}^2U_n+3abcU_{n+1}U_n^2+c(b^2-2ac)U_n^3 =$$
$$= a(aU_{n+1}^2-cU_n^2)(bU_{n+1}+2cU_n)+cU_n(aU_{n+1}+bU_n)(bU_n+4aU_{n+1})-abcU_{n+1}U_n)$$
$$= ab(aU_{n+1}U_{2n+1}-cU_nU_{2n})+2c^2U_nU_{n-1}(2aU_{n+1}+bU_n) =$$
$$= a^2(bU_{3n+1}+2cU_{3n}) = -a^3V_{3n-1}$$

and

$$2a^3U_{n+1}^3+3a^2bU_{n+1}^2U_n+3a(b^2-2ac)U_{n+1}U_n^2+b(b^2-3ac)U_n^3 =$$
$$= a(aU_{n+1}^2-cU_n^2)(2aU_{n+1}+bU_n)+U_n(aU_{n+1}+bU_n)(2abU_{n+1}+b^2U_n-2acU_n)-$$
$$-2ac^2U_{n+1}U_n^2 =$$
$$= a^2(2aU_{3n+1}+bU_{3n}) = a^3V_{3n}$$

so that

$$S(U_{3n+1}/U_{3n}) = V_{3n+1}/V_{3n}.$$

Taking $\bar{x}_0^* = 1$ from here we find the subsequence $\{\bar{x}_n^*\}$ with $\bar{x}_n^* = V_{3^n+1}/V_{3^n}$.

R E F E R E N C E S

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НЕКОИ ЗАБРЗУВАЊА НА КОНВЕРГЕНЦИЈАТА
НА ОПРЕДЕЛЕНИ КЛАСИ ОД НИЗИ

Р е з и м е

Низата $\{U_{n+1}/U_n\}$ од количниците на последователните броеви U_n, $n=0,1,2,\ldots$ определени со $aU_{n+1}+bU_n+cU_{n-1} = 0$ и $U_0 = 0$, $U_1 = 1$ конвергираат кон коренот λ_1 на $f(x)=ax^2+bx+c = 0$, при претпоставка $|\lambda_1| > |\lambda_2|$. Се покажува дека со методот на Newton за равенката $f(x)=0$ и почетно приближување 1 се добива подниза та $\{U_{2^n+1}/U_{2^n}\}$, додека со методот на итерација на Holley се добива поднизата $\{U_{3^n+1}/U_{3^n}\}$.