ON A CLASS OF THE SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract: The objective is to show the analogy between a theorem of M. Petrovitch and a result of Moutard concerning the differential equation of the second order.

1. M. Petrovitch [1] has proven the following theorem: for the integrable equation

$$y' + y^2 = f(x)$$

it may be attached a series of functions $\mu_i(x)$, $i = 1, 2, \ldots$ so that every equation

$$y_i' + y_i^2 = f(x) + \mu_i(x) \tag{1}$$

without any additional quadrature is integrable too.

The proof is based on formation of a series of the recurrent formulas of the form

$$X_n = X_{n-1} + \frac{3}{4}\left(\frac{X'_{n-1}}{X_{n-1}}\right)^2 - \frac{1}{2}\,\frac{X''_{n-1}}{X_{n-1}}, \quad n = 1, 2, \ldots \tag{2}$$

and taking

$$\mu_i(x) = X_i - X_{i-1} \tag{3}$$

with $X_0 = f(x)$.

The purpose of this note is to show the analogy between the theorem of M. Petrovitch above mentioned and a result of Moutard [2] concerning a differential equation of the second order.

2. The Moutard equation

$$y'' + \left(2h + \frac{d}{dx}\lg\omega\right) y' - \lambda y = 0 \tag{4}$$

where $\omega = \omega(x)$, $\lambda = \lambda(x)$ and $h =$ const., by the successive transformations

$$y'_{i-1} = \lambda_{i-1} y_i, \quad i = 1, 2, \ldots$$

with $y_0 = y$ and $\lambda_0 = \lambda$, produces the equations

$$y_i'' + \left(2h + \frac{d}{dx}\lg\omega\,\lambda\,\lambda_i \ldots \lambda_{i-1}\right) y_i' - \lambda_i y_i = 0 \tag{5}$$

where

$$\lambda_i = \lambda_{i-1} - \frac{d^2}{dx^2}\lg\omega\,\lambda\,\lambda_i \ldots \lambda_{i-1} \quad i = 1, 2, \ldots$$

By the substitutions

$$y_i = e^{-hx}\left(\omega \lambda \lambda_i \ldots \lambda_{i-1}\right)^{-1/2} \exp\int u_i(x)dx$$

in (5) we get the Riccati form of these equations

$$u_i' + u_i^2 = \lambda_{i-1} + h\left(h + \frac{d}{dx}\lg \omega \lambda \lambda_i \ldots \lambda_{i-1}\right) + \frac{1}{4}\left(\frac{d}{dx}\lg \omega \lambda \lambda_i \ldots \lambda_{i-1}\right)^2 - \frac{1}{2}\frac{d^2}{dx^2}\lg \omega \lambda \lambda_i \ldots \lambda_{i-1} \quad (6)$$

or

$$u_i' + u_i^2 = \lambda_{i-1} + h\left(h + \frac{(\omega \lambda \lambda_1 \ldots \lambda_{i-1})'}{\omega \lambda \lambda_1 \ldots \lambda_{i-1}}\right) + \frac{3}{4}\left[\frac{\omega'}{\omega} + \frac{\lambda'}{\lambda} + \frac{\lambda_1'}{\lambda_1} + \ldots + \frac{\lambda_{i-1}'}{\lambda_{i-1}}\right]^2 - \frac{1}{2}\frac{(\omega \lambda \lambda_1 \ldots \lambda_{i-1})''}{\omega \lambda \lambda_i \ldots \lambda_{i-1}}$$

Let us give following M. Petrovitch, to the series of equations (1) with (2) and (3) the form

$$y_i' + y_i^2 = f(x) + \frac{1}{4}\left[\left(\frac{d}{dx}\lg x_0\right)^2 + \left(\frac{d}{dx}\lg x_1\right)^2 + \ldots + \left(\frac{d}{dx}\lg x_{i-1}\right)^2\right] - \frac{1}{2}\frac{d^2}{dx^2}\lg x_0 x_1 \ldots x_{i-1} \quad (7)$$

or

$$y_i' + y_i^2 = f(x) + \frac{3}{4}\left[\left(\frac{x_0'}{x_0}\right)^2 + \left(\frac{x_1'}{x_1}\right)^2 + \ldots + \left(\frac{x_{i-1}'}{x_{i-1}}\right)^2\right] - \frac{1}{2}\left[\frac{x_0''}{x_0} + \frac{x_1''}{x_1} + \ldots + \frac{x_{i-1}''}{x_{i-1}}\right]$$

It is clear that for $h = 0$ and $\omega = \text{const.}$, we have an analogy between (6) and (7).

Moutard notified that for $\lambda_i = 0$ we obtain from (5) y_i by the quadrature. Consequently, we get μ_i by the differentiation.

3. We note that the classes of equations of M. Petrovitch and Moutard (special case) arise from the equation

$$y'' = f(x)\,y$$

which enables unification of their results.

Indeed, by the elementary transformation [4] $y = f^{k}u$, $f = f(x)$, $u = u(x)$, $k \in R$, we have

$$u'' + u'\frac{d}{dx}\lg f^{2k-1} + \left[k(k-2)\frac{f'^2}{f^2} + k\frac{f''}{f} - f\right] u = 0 \qquad (8)$$

which is a special case of Moutard equation.

For $k = ½$ it is

$$u'' = X_1 u$$

and after differentiation, by the transformation $u' = \sqrt{X_1}\ , u_1$ gets the form

$$u_1'' = X_2 u_1$$

where X_1, X_2 are given by (2), and $u_1 = u_1(x)$.

The successive repeating of the procedure by $u_{i-1}' = \sqrt{X_i}\, u_i$, $i = 2, 3, \ldots$ yields

$$u_i'' = X_{i+1} u_i, \qquad i = 2, 3, \ldots$$

whose Riccati form is given by (7).

If $k = 1$, from (8) we get the Moutard form (4) with $h = 0$, $\omega = f$ and $\lambda = f - \frac{d^2}{dx^2}\lg f$.

REFERENCES

[1] M. Petrovitch, *Théoreme sur l'equation de Riccati*, Publication mathématique de l'Universite de Beigrade, Belgrad, t. IV, (1935), p.169.

[2] H. Laurent, *Traite d'Analyse*, Paris, t. V, 1890.

[3] M. Moutrard, *Comptes Rendus*, Paris, 1875.

[4] B. S. Popov, *Analogy of one theorem of M. Petrovitch*, Contributions, Sec. for Nat. Sc. and Math., Skopje, 1969, Vol. 1, p. 5.

Резиме

ЗА ЕДНА КЛАСА ЛИНЕАРНИ ДИФЕРЕНЦИЈАЛНИ РАВЕНКИ ОД ВТОР РЕД

Се покажува аналогијата меѓу една теорема на M. Perovitch и еден резултат на Moutard, која се однесува за диференцијални равенки од втор ред.