SUMMATION OF RECIPROCAL SERIES OF A CLASS OF POLYNOMIALS

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Abstract: Several relations concerning the sums of reciprocal series of a certain class of polynomials are given.

1. This note deals with the class of polynomials $W_n(p,q;k,x)$ defined by [1]

$$W_n(x) = kxW_{n-1}(x) + W_{n-2}(x), \quad n \geq 2, \quad k = 1, 2$$

$$W_0 = p, W_1 = q, \text{ with } W_n(x) = W_n(p,q;k,x).$$

We are interested in the polynomials $W_n(0,1;1,x) = F_n(x)$ and $W_n(2,x;1,x) = L_n(x)$ known as Fibonacci and Lucas polynomials respectively, and $W_n(0,1;2,x) = P_n(x)$ and $W_n(2,2x;2,x) = Q_n(x)$ named Pell and Pell-Lucas polynomials respectively.

These polynomials are introduced also by the relation [2]

$$F_n(x) = \left(\alpha^n - \beta^n\right)/(\alpha - \beta), \quad n \geq 0$$

$$L_n(x) = \alpha^n + \beta^n, \quad n \geq 0$$

where

$$\alpha = \left(x + \sqrt{x^2+4}\right)/2, \quad \beta = \left(x - \sqrt{x^2+4}\right)/2 \tag{1}$$

and

$$P_n(x) = \left(a^n - b^n\right)/(a-b), \quad n \geq 0$$

$$Q_n(x) = a^n + b^n, \quad n \geq 0$$

where

$$a = x + \sqrt{x^2+1}, \quad b = x - \sqrt{x^2+1} \tag{2}$$

The purpose of this note is to give some relations concerning the sums of reciprocal series of these polynomials. It is an extension of the result of H-J. Seiffert obtained for sequences of numbers [3]

2. Let

$$\sqrt{x^2+4}\; F_m(x)Q_m(x) = A_m(x) = A_m,$$

$$\sqrt{x^2+1}\; L_m(x)P_m(x) = B_m(x) = B_m.$$

We have

$$\frac{2B_m}{B_m^2 - A_m^2} = \frac{1}{B_m - A_m} + \frac{1}{B_m + A_m} \tag{3}$$

$$\frac{2A_m}{B_m^2 - A_m^2} = \frac{1}{B_m - A_m} - \frac{1}{B_m + A_m}. \tag{4}$$

But

$$B_m - A_m = \left(\alpha^m + \beta^m\right)\left(a^m - b^m\right) - \left(\alpha^m - \beta^m\right)\left(a^m + b^m\right)$$
$$= 2\left(\beta^m a^m - \alpha^m b^m\right) = 2\left(1 - (\alpha b)^{2m}\right)\big/(\alpha b)^m ,$$

$$B_m + A_m = \left(\alpha^m + \beta^m\right)\left(a^m - b^m\right) + \left(\alpha^m - \beta^m\right)\left(a^m + b^m\right)$$
$$= 2\left(\alpha^m a^m - \beta^m b^m\right) = 2\left(1 - (\beta b)^{2m}\right)\big/(\beta b)^m .$$

Then

$$\sum_{m=1}^{n} \frac{A_m}{B_m^2 - A_m^2} = \frac{1}{4} \sum_{m=1}^{n} \left(\frac{(\alpha b)^m}{1-(\alpha b)^{2m}} - \frac{(\beta b)^m}{1-(\beta b)^{2m}} \right)$$

$$\sum_{m=1}^{n} \frac{B_m}{B_m^2 - A_m^2} = \frac{1}{4} \sum_{m=1}^{n} \left(\frac{(\alpha b)^m}{1-(\alpha b)^{2m}} + \frac{(\beta b)^m}{1-(\beta b)^{2m}} \right).$$

From the identity [2]

$$\sum_{k=1}^{n} \frac{z^{2^{k-1}}}{1 - z^{2^k}} = \frac{z}{1-z} \frac{z - z^{2^n}}{1 - z^{2^n}}$$

putting $z = \alpha b$ and $z = \beta b$ respectively, it is

$$\sum_{n=1}^{\infty} \frac{(\alpha b)^{2^n}}{1-(\alpha b)^{2^{n+1}}} = \frac{(\alpha b)^2}{1-(\alpha b)^2} , \tag{5}$$

and

$$\sum_{n=1}^{\infty} \frac{(\beta b)^{2^n}}{1-(\beta b)^{2^{n+1}}} = \frac{(\beta b)^2}{1-(\beta b)^2} \tag{6}$$

taking that $|\alpha b| < 1$ and $|\beta b| < 1$.

Now from (3), (4) and (5), (6) we obtain

$$\sum_{n=1}^{\infty} \frac{F_{2^n}(x) Q_{2^n}(x)}{4\left(x^2+1\right)\left(L_{2^n}(x)P_{2^n}(x)\right)^2 - \left(x^2+4\right)\left(F_{2^n}(x)Q_{2^n}(x)\right)^2}$$
$$= \frac{1}{4\sqrt{x^2+4}} \left(\frac{(\alpha b)^2}{1-(\alpha b)^2} - \frac{(\beta b)^2}{1-(\beta b)^2} \right)$$
$$= \frac{b^2}{4} \frac{F_2(x)}{1 - b^2 L_2(x) + \left(\alpha\beta b^2\right)^2} = \frac{F_2(x)}{4\left[Q_2(x) - L_2(x)\right]}$$

and

$$\sum_{n=1}^{\infty} \frac{L_{2^n}(x) P_{2^n}(x)}{4\left(x^2+1\right)\left(L_{2^n}(x)P_{2^n}(x)\right)^2 - \left(x^2+4\right)\left(F_{2^n}(x)Q_{2^n}(x)\right)^2}$$

$$= \frac{1}{8\sqrt{x^2+1}}\left(\frac{(\alpha b)^2}{1-(\alpha b)^2} + \frac{(\beta b)^2}{1-(\beta b)^2}\right)$$

$$= \frac{b^2}{8\sqrt{x^2+1}} \frac{L_2(x) - 2(\alpha\beta b)^2}{1 - L_2(x)b^2 + (\alpha\beta b^2)^2} = \frac{L_2(x) - 2b^2}{8\sqrt{x^2+1}(Q_2(x) - L_2(x))}.$$

Putting α, β, a and b from (1) and (2) we find

$$\sum_{n=1}^{\infty} \frac{F_{2^n}(x)Q_{2^n}(x)}{4(x^2+1)(L_{2^n}(x)F_{2^n}(x))^2 - (x^2+4)(F_{2^n}(x)Q_{2^n}(x))^2} = \frac{1}{12x} \quad (7)$$

and

$$\sum_{n=1}^{\infty} \frac{L_{2^n}(x)P_{2^n}(x)}{4(x^2+1)(L_{2^n}(x)P_{2^n}(x))^2 - (x^2+4)(F_{2^n}(x)Q_{2^n}(x))^2} = \frac{1}{6x} - \frac{1}{8\sqrt{x^2+1}}. \quad (8)$$

3. As a consequence from (7) and (8) we have

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{x^2+1}\ L_{2^n}(x)P_{2^n}(x) \mp \sqrt{x^2+4}F_{2^n}(x)Q_{2^n}(x)} = \frac{\sqrt{x^2+1}}{3x} - \frac{x}{4} \pm \frac{\sqrt{x^2+4}}{12x}.$$

If we take x=1, we obtain from (7) and (8)

$$\sum_{n=1}^{\infty} \frac{F_{2^n}Q_{2^n}}{8(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} = \frac{1}{12} \quad (9)$$

and

$$\sum_{n=1}^{\infty} \frac{L_{2^n}P_{2^n}}{8(L_{2^n}P_{2^n})^2 - 5(F_{2^n}Q_{2^n})^2} = \frac{1}{6} - \frac{1}{8\sqrt{2}}, \quad (10)$$

where $F_n(1) = F_n$, $L_n(1) = L_n$, $P_n(1) = P_n$ and $Q_n(1) = Q_n$.

From (9) and (10) it is

$$\sum_{n=1}^{\infty} \frac{1}{2\sqrt{2}L_{2^n}P_{2^n} \mp \sqrt{5}F_{2^n}Q_{2^n}} = \frac{\sqrt{2}}{3} - \frac{1}{4} \pm \frac{\sqrt{5}}{12}.$$

REFERENCES

[1] G. Palama, Contributo alla ricerca di relazioni fra classici polinomi, Revisita di Matematica dell l'Universita di Parma 2, p. 383–402, 1959.

[2] E. Lucas, Théorie des nombres, Paris 1890.

[3] H.-J. Seiffert, Advanced Problems, Fib. Quart. Vol. 34, No. 4, 1994.

Резиме

СУМИРАЊЕ НА РЕЦИПРОЧНИ РЕДОВИ ОД ЕДНА КЛАСА НА ПОЛИНОМИ

Во трудот се даваат неколку релации кои се однесуваат за сумите од реципрочни редови на една класа полиноми.