A NOTE ON A CLASS OF POLYNOMIALS

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Abstract

The class of polynomials whose argument is the polynomial of the same kind are considered.

1. In an earlier paper G. Palama [1] has considered the analogy between the Cauchy polynomials

$$\psi_n(x) = 2n \sum_{k=0}^{n} \frac{(2n-k-1)!}{k!\,(2n-2k)!}\,\frac{x^{n-k}}{2^{2k}}, \quad \psi_0(x) = 2$$

and the Lucas polynomials $V_n(x, p)$ defined by

$$V_n(x, p) = \alpha^n(x) + \beta^n(x), \quad n \ge 0 \tag{1}$$

where

$$\alpha(x) = \left(x + \sqrt{x^2 - 4p}\right)/2, \qquad \beta(x) = \left(x - \sqrt{x^2 - 4p}\right)/2.$$

He shows that the polynomials $\psi_n(x)$ may be given by the recurrent formula

$$\psi_{n+1}(x) = \left(x + \frac{1}{2}\right)\psi_n(x) - \frac{1}{2^4}\,\psi_{n-1}(x), \quad \psi_0(x) = 2, \quad \psi_1(x) = x + \frac{1}{2}. \tag{2}$$

The purpose of this note is to give the form of the companion polynomials $\varphi_n(x)$ and some characteristics of the polynomials $\varphi_n(x)$ and $\psi_n(x)$, by an elementary method.

2. By the relation (2) we have

$$\psi_n(x) = \sum_{k=c}^{n} \frac{n}{n-k}\binom{2n-k-1}{k}\frac{x^{n-k}}{2^{2k}}, \quad \psi_0(x) = 2.$$

Let us define the polynomials $\varphi_n(x)$ by the recurrent formula

$$\varphi_{n+1}(x) = \left(x + \frac{1}{2}\right)\varphi_n(x) - \frac{1}{2^4}\,\varphi_{n-1}(x), \quad \varphi_0(x) = 0, \quad \varphi_1(x) = 1$$

from where we have

$$\varphi_{n+1}(x) = \sum_{k=0}^{n}\binom{2n-k+1}{k}\frac{x^{n-k}}{2^{2k}}, \quad \varphi_0(x) = 0.$$

The definition (1) leads to the relation

$$V_{2n}\left(2\sqrt{x}, -p\right) = (4p)^n\, V_n\left(\frac{x}{p} + \frac{1}{2}, \frac{1}{2^4}\right) \tag{3}$$

and comparing with (2) we have

$$V_n\left(x + \frac{1}{2}, \frac{1}{2^4}\right) = \psi_n(x). \tag{4}$$

By the property [2] of the polynomils $V_n(x, p)$, that every formula containing x, p, $V_n(x, p)$ may be generalised by the substitution of x with $V_r(x, p)$, p with p^r and $V_n(x, p)$ with $V_{rn}(x, p)$, from (3) we obtain

$$V_{2n}(x,p) = (-1)^n 2^{2n} p^n V_n\left(-\frac{x^2}{4p}, \frac{1}{2^4}\right)$$

and consequently, taking that $V_r^2(x,p) = V_{2r}(x,p) + 2p^r$, we have

$$V_{2nr}(x,p) = (-1)^n 2^{2n} p^{nr} V_n\left(-\frac{V_{2r}(x,p)}{4p^r}, \frac{1}{2^4}\right). \tag{5}$$

The relations (3) and (4) give

$$\psi_n\left(\frac{x}{p}\right) = (4p)^{-n} V_{2n}\left(2\sqrt{x}, -p\right).$$

Then from (5) we have

$$\psi_{rn}\left(\frac{x}{p}\right) = \left(-2^{-2}\right)^{(r-1)n} \psi_n\left(\left(-2^2\right)^{r-1} \psi_r\left(\frac{x}{p}\right) - \frac{1}{2}\right)$$

$r = 2^m$, $m = 0, 1, 2, \ldots$

3. The analogy established between the polynomials $V_n(x,p)$ and $\psi_n(x)$ leads to the polynomials $U_n(x,p)$ defined by

$$U_n(x,p) = \left(\alpha^n(x) - \beta^n(x)\right) / \left(\alpha(x) - \beta(x)\right) \tag{6}$$

which are the generalisation of the polynomials $\varphi_n(x)$, and to the relation

$$\frac{d\psi_n(x)}{dx} = n\varphi_n(x).$$

From (5) and (6) we find

$$U_{2n}\left(2\sqrt{x}, -p\right) = 2(4p)^{n-1}\sqrt{x}\,\varphi_n\left(\frac{x}{p}\right) \tag{7}$$

and

$$\varphi_n(x) = U_n\left(x + \frac{1}{2}, \frac{1}{2^4}\right). \tag{8}$$

Consequently by (7) and (8) we have

$$\varphi_{rn}\left(\frac{x}{p}\right) = \left(-2^{-2}\right)^{(r-1)(n-1)} \varphi_n\left(\left(-2^2\right)^{r-1} \psi_r\left(\frac{x}{p}\right) - \frac{1}{2}\right) \varphi_r\left(\frac{x}{p}\right)$$

$r = 2^m$, $m = 0, 1, 2$.

References

[1] Palama, G.: *Contributo alla ricarea di relationi fra classici polinomi*, Revista di Matematica dela Universita di Parma 2, 383–462 (1951).

[2] Lucas, E.: *Theorie des nombres t.* 1 Paris (1891).

ЗАБЕЛЕШКА ЗА ЕДНА КЛАСА ПОЛИНОМИ

Резиме

Се разгледува аналогијата меѓу полиномите на Cauchy и тие на Lucas, кои всушност се нивна генерализација.