ON RECIPROCAL SERIES RELATED TO TCHEBYCHEFF POLYNOMIALS

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A b s t r a c t: Some reciprocal series of Tchebycheff polynomials are given.

1. We consider the Tchebycheff polynomials $T_n(x)$ and $U_n(x)$ of the first and second kinds respectively defined by

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),\ n \geq 1,\ T_0(x) = 1,\ T_1(x) = x$$

and

$$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x),\ n \geq 1,\ U_0(x) = 0,\ U_1(x) = 1$$

They can be expressed in the form

$$(1)\quad 2T_n(x) = \alpha^n(x) + \beta^n(x)$$

and

$$(2)\quad \Delta(x)U_n(x) = \alpha^n(x) - \beta^n(x)$$

where

$$\alpha(x) = x + \sqrt{x^2 - 1},\ \beta(x) = x - \sqrt{x^2 - 1},\ \delta(x) = \alpha(x) - \beta(x) = \sqrt{\Delta(x)}.$$

The definition may be extended by $T_{-n}(x) = T_n(x)$ and $U_{-n}(x) = - U_n(x)$.

The purpose of this note is to give some relations concerning the reciprocal series of these polynomials.

2. Using (1) and (2) we obtain

$$2\alpha^n(x) = 2T_n(x) - \delta(x)\ U_n(x)$$

and

$$4\alpha^{m+n}(x) = 4T_n(x)\ T_m(x) + \Delta(x)\ U_n(x)\ U_m(x) + 2\delta(x)\ (T_n(x)\ U_m(x) + T_m(x)\ U_n(x))$$

from which it follows that

$$(3)\qquad U_{r+s}(x)\ T_s(x) - T_{r+s}(x)\ U_s(x) = U_r(x)$$

and

$$(4)\qquad 4T_{r+s}(x)\ T_s(x) - \Delta(x)\ U_{r+s}(x)\ U_s(x) = 4T_r(x)$$

If we write (3) in the form

$$\frac{T_s(x)}{U_s(x)} - \frac{T_{r+s}(x)}{U_{r+s}(x)} = \frac{U_r(x)}{U_s(x)U_{r+s}(x)}$$

and replace s here by s, s + r, s + 2r, ... s + (n − 1)r successively and add the results, we obtain

$$(5)\qquad S_n^{r,s} = \sum_{k=1}^{n} \frac{1}{U_{s+(k-1)r}U_{s+kr}} = \left(\frac{T_s(x)}{U_s(x)} - \frac{T_{s+nr}(x)}{U_{s+nr}(x)}\right)\frac{1}{U_2(x)} = \frac{U_{nr}(x)}{U_s(x)U_r(x)U_{s+nr}(x)}$$

Similarly, using (3) again we have

$$(6)\qquad \sigma_n^{r,s}(x) = \sum_{k=1}^{n} \frac{1}{T_{s+(k-1)}(x)T_{s+kr}(x)} = \left(\frac{U_{s+nr}(x)}{T_{s+nr}(x)} - \frac{U_s(x)}{T_s(x)}\right)\frac{1}{U_2(x)} = \frac{U_{nr}(x)}{T_s(x)\ U_r(x)\ T_{s+nr}(x)}$$

Because

$$\lim_{n \to \sim} \frac{U_n(x)}{U_{n+s}(x)} = \begin{cases} \beta^s(x), \left|\frac{\beta(x)}{\alpha(x)}\right| < 1, & x > 1 \\ \alpha^s(x), \left|\frac{\alpha(x)}{\beta(x)}\right| < 1, & x < -1 \end{cases}$$

and

$$\lim_{n \to \sim} \frac{U_n(x)}{T_{s+n}(x)} = \begin{cases} \frac{\beta^s(x)}{\delta}, & x > 1 \\ \frac{\alpha^s}{\delta}, & x < -1 \end{cases}$$

we find

(7) $$S^{R,S}(x) = \sum_{n=1}^{\infty} \frac{1}{U_{s+(n-1)r}(x)U_{s+nr}(x)} = \begin{cases} \frac{\beta^s(x)}{U_r(x)U_s(x)}, & x > 1 \\ \frac{\alpha^s(x)}{U_s(x)U_r(x)}, & x < -1 \end{cases}$$

and

(8) $$\sigma^{r,s}(x) = \sum_{n=1}^{\infty} \frac{1}{T_{s+(n-1)r}(x)T_{s+n}(x)} = \begin{cases} \frac{\beta^s(x)}{\delta(x)T_s(x)U_r(x)}, & x > 1 \\ \frac{\alpha^s(x)}{\delta(x)T_s(x)U_r(x)}, & x < -2 \end{cases}$$

In particular, if r = s we have

(9) $$S^{r,r}(x) = \sum_{n=1}^{\infty} \frac{1}{U_{kr}(x)U_{(n+1)r}(x)} = \begin{cases} \frac{\beta^r(x)}{U_r^2(x)}, & x > 1 \\ \frac{\alpha^r(x)}{U_r^2(x)}, & x < -1 \end{cases}$$

and

(10) $$\sigma^{r,r}(x) = \sum_{n=1}^{\infty} \frac{1}{T_{nr}(x)T_{(n+1)r}(x)} = \begin{cases} \frac{2\beta_r^r(x)}{U_r^2(x)}, & x > 1 \\ \frac{2\alpha^r(x)}{U^2{}_{2r}(x)}, & x < -1 \end{cases}$$

3. Following the method used above from the relation

$$2\,(T_{2s}(x) - T_{2r}(x)) = \Delta(x)\, U_{r+s}(x)\, U_{r-s}(x)$$

we obtain

(11) $$\Delta(x) \sum_{k=1}^{n} \frac{1}{T_{2r+2(2k-1)s}(x) - T_{2s}} = \frac{2U_{2ns}}{U_2(x)U_{2s}(x)U_{r+2ns}(x)}$$

and

(12) $$\Delta(x)\sum_{n=1}^{\infty}\frac{1}{T_{2r+2(2n-1)s}(x)-T_{2s}(x)}=\begin{cases}\dfrac{\beta^r(x)}{U_r(x)\;U_{2s}(x)}, & x>1\\ \dfrac{\alpha^r(x)}{U_r(x)\;U_{2s}(x)}, & x<-1\end{cases}$$

Similarly, from the relation

$$2\left(T_{r+s}(x)+T_{r-s}(x)\right)=T_{2r}(x)+T_{2s}(x)$$

we obtain

(13) $$\sum_{k=1}^{n}\frac{1}{T_{2r+2(k-1)s}(x)-T_{2s}(x)}=\frac{U_{2ks}}{U_{rs}(x)T_r(x)T_{r+2ns}}$$

from which

(14) $$\sum_{k=1}^{\infty}\frac{1}{T_{2r+2(k-1)s}(x)+T_{2s}}=\begin{cases}\dfrac{\beta^r(x)}{U_{rs}(x)\;T_r(x)}, & x>1\\ \dfrac{\alpha^r(x)}{U_{rs}(x)\;T_2(x)}, & x<-2\end{cases}$$

We notice that from

$$U_r^2(x)-U_s^2(x)=U_{r+s}(x)U_{r-s}(x)$$

it follows that

(15) $$\sum_{k=1}^{n}\frac{1}{U^2_{rs+(k-1)r}(x)-U_r^2}=S_n^{r,s}(x)$$

Similarly, from

$$T_r^2(x)-T_s^2(x)=T_{r+s}(x)\quad T_{r-\delta}(x)$$

we obtain

(16) $$\Delta(x)\sum_{k=1}^{n}\frac{1}{T^2_{2s+(2k-1)s}(x)-T_r^2(x)}=\delta_n^{r,s}(x)$$

REFERENCES

1. E. D. Rainville: *Special Functions*, New York 1960.
2. H. Batemann: *Higer Transeedental Functions*, vol. II, New York, 1955.

Резиме

РЕЦИПРОЧНИ РЕДОВИ ЗА ПОЛИНОМИТЕ НА ЧЕБИШЕВ

Се даваат некои елементарни редови за полиномите на Чебишев.