EXPRESSIONS OF JACOBI POLYNOMIALS THROUGH BERNOULLI POLYNOMIALS

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A b s t r a c t: One classical technique for expanding the Jacobi polynomial in a series of Bernoulli polynomials is presented.

1. *Introduction*

There are numerous relations interconnecting classicial polynomials mutually. One kind of these relations uses the property that a simple set of polynomials may be expanded in a series of polynomials. This becomes particularly pleasent, when one has to deal with an orthogonal set [1], [2].

Following the method which uses the classical technique to expand a polynomial in a series of other class of polynomials we obtain the expansions of Jacobi, Gegenbauer and Legendre polynomials through Bernoulli polynomials.

2. *Preliminaires*

The Bernoulli polynomial $B_n(x)$ is defined by the generating relation [3]

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{\infty}B_n(x)\frac{t^n}{n!}. \tag{1}$$

Because

$$\frac{te^{tx}}{e^t-1}=\frac{t}{e^t-1}e^{tx}=\sum_{r=0}^{\infty}B_r\frac{t^r}{r!}\sum_{m=0}^{\infty}\frac{(tx)^m}{m!}$$

where B_r are Bernoulli numbers, we conclude that

$$B_r(x)=\sum_{r=0}^{n}\binom{n}{r}B_rx^{n-r}. \tag{2}$$

It is evident that $B_r(0)=B_r$.

From (1) we have

$$\sum_{n=0}^{\infty}(B_{n+1}(x+1)-B_n(x))\frac{t^n}{n!}=te^{tx}=\sum_{n=0}^{\infty}\frac{x^{n-1}}{(n-1)!}t^n$$

and we obtain

$$B_n(x+1)-B_n(x)=nx^{n-1},\qquad n=0,1,2,... \tag{3}$$

Since

$$\sum_{n=0}^{\infty}B_n(x+1)\frac{t^n}{n!}=\frac{te^{t(x+1)}}{e^t-1}=\frac{te^{tx}}{e^t-1}e^t=\sum_{r=0}^{\infty}B_r(x)\frac{t^r}{r!}\sum_{m=0}^{\infty}\frac{t^m}{m!}$$

after multiplying and comparing the coefficients, we have

$$\sum_{r=0}^{n}\binom{n}{r}B_r(x)=B_{n+1}(x+1).$$

From (2) and (3) we obtain

$$x^n=\sum_{r=0}^{n}\frac{n!B_r(x)}{r!(n-r+1)!} \tag{4}$$

This formula simplifies the procedure of expanding the polynomials in a series of Bernoulli polynomials.

Key words: Jacobi polynomial, Bernoulli polinomial

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ with $\alpha > -1$, $\beta > -1$ is defined by [2]

$$P_n^{(\alpha,\beta)}(x) = \binom{n+\alpha}{n}\, {}_2F_1(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}) \qquad (5)$$

where

$${}_2F_1(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!}$$

and

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), \quad n \ge 1$$

$$(\alpha)_0 = 1, \quad \alpha \ne 0,$$

are the hypergeometric function and the Pochhammer symbol respectively. The generalised hypergeometric function ${}_pF_q\,(\alpha_1, \alpha_2, ..., \alpha_p; \beta_1, \beta_2, ..., \beta_q; x)$ which is used later is similarly defined.

The equation (5) after replacing x by $(-x)$ yields expanded form for $P_n^{(a,\beta)}(x)$, namely

$$P_n^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} \frac{(-1)^{n+k}(1+\beta)_n (1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\beta)_k (1+\alpha+\beta)_n} \left(\frac{x+1}{2}\right)^k,$$

which can be written in the form

$$\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(2x-1)}{(1+\beta)_n} = \sum_{s=0}^{n} \frac{(-1)^{n+s}(1+\alpha+\beta)_{n+s} x^s}{s!(n-s)!(1+\beta)_s}. \qquad (6)$$

3. *The relation between Jacobi and Bernoulli polynomials*

Consider the series

$$\psi(x,t) = \sum_{n=0}^{\infty} \frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(2x-1) t^n}{(1+\beta)_n}. \qquad (7)$$

Using (6) and (7), we obtain

$$\psi(x,t) = \sum_{n=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^{n+s}(1+\alpha+\beta)_{n+s} x^s t^n}{s!(n-s)!(1+\beta)_s}$$

$$= \sum_{n,s=0}^{\infty} \frac{(-1)^n (1+\alpha+\beta)_{n+2s} x^s t^{n+s}}{s!\,n!(1+\beta)_s},$$

where the identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,n) = \sum_{n,k=0}^{\infty} A(k, n+k) \qquad (8)$$

has been used to collect the powers of t in the last summation above.

Hence by (4) we may write

$$\psi(x,t) = \sum_{n,s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^n (1+\alpha+\beta)_{n+2s} B_k(x) t^{n+s}}{n!(1+\beta)_s\, k!(s-k+1)!}$$

$$= \sum_{n,k,s=0}^{\infty} \frac{(-1)^n (1+\alpha+\beta)_{n+2s+2k} B_k(x) t^{n+s+k}}{n!\,k!(s+1)!\,(1+\beta)_{s+k}}$$

in which we have used again the identity (8).

By the identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k, n-k) \qquad (9)$$

the reverse of (8), we write

$$\psi(x,t)=\sum_{n,k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^{n-s}(1+\alpha+\beta)_{n+s+2k}B_k(x)t^{n+k}}{(n-s)!\,k!\,(s+1)!\,(1+\beta)_{s+k}}$$

$$=\sum_{n,k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^n(1+\alpha+\beta+n+2k)_s(-n)_s(1)_s(1+\alpha+\beta)_{n+2k}B_k(x)t^{n+k}}{s!n!(2)_s k!\,(1+\beta+k)_s(1+\beta)_k}$$

$$=\sum_{n,k=0}^{\infty}{}_3F_2\left(\begin{matrix}-n,1+n+2k+\alpha+\beta,1\\1+k+\beta,2\end{matrix}\;1\right)\frac{(-1)^n(1+\alpha+\beta)_{n+2k}B_k(x)t^{n+k}}{n!\,k!\,(1+\beta)_k}.$$

Using again the identity (9) we have

$$\psi(x,t)=\sum_{k=0}^{\infty}\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,1+n+k+\alpha+\beta,1\\1+\beta+k,2\end{matrix}\;1\right)\frac{(-1)^{n-k}(1+\alpha+\beta)_{n+k}B_k(x)t^n}{(n-k)!\,k!\,(1+\beta)_k}.$$

Finaly we may conclude that

$$P_n^{(\alpha,\beta)}(2x-1)=\frac{(1+\beta)_n}{(1+\alpha+\beta)_n}\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,1+\alpha+\beta+n+k,1\\1+\beta+k,2\end{matrix}\;1\right)\cdot \frac{(-1)^{n-k}(1+\alpha+\beta)_{n+k}B_k(x)}{(n-k)!\;k!\;(1+\beta)_k}. \tag{10}$$

4. *Special cases*

1°. If $\alpha=\beta$, the Jacobi polynomial is called an ultraspherical polynomial (termed also Gegenbauer – $C_n^\nu(x)$). They are essentially equivalent i.e.

$$C_n^\nu(x)=\frac{(2\nu)_n P_n^{(\nu-\frac{1}{2},\nu-\frac{1}{2})}(x)}{(\nu+\frac{1}{2})_n},\qquad P_n^{(\alpha,\alpha)}(x)=\frac{(1+\alpha)_n C_n^{\alpha+\frac{1}{2}}(x)}{(1+2\alpha)_n}.$$

From (10) we obtain

$$C_n^\nu(2x-1)=\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,n+2\nu+k,1\\\nu+\frac{1}{2}+k,2\end{matrix}\;1\right)\frac{(-1)^{n-k}(2\nu)_{n+k}B_k(x)}{k!\,(n-k)!(\nu+\frac{1}{2})_k}. \tag{11}$$

2°. If $\alpha=\beta=0$, the polynomial (6) becomes the Legendre polynomilal $P_n^{(0,0)}(x)=P_n(x)$ and consequently we have the relation

$$P_n(2x-1)=\sum_{k=0}^{n}{}_3F_2(-n+k,1+n+k,1;k+1,2;1)\,\frac{(-1)^{n-k}(n+k)!B_k(x)}{(n-k)!\,k!\,k!}. \tag{12}$$

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[2] Szego G.: *Orthogonal Polynomials*, American Mathematical Society, New York, 1950.

[3] Erdelyi A., et al.: *Higher Transedental Functions*, Vol. 1. Mc-Graw-Hill, New York – Toronto – London, 1952.

Резиме

ПРЕТСТАВУВАЊЕ НА ЈАКОБИЕВИ ПОЛИНОМИ СО БЕРНУЛИЕВИ ПОЛИНОМИ

Во трудот е прикажана една метода за развивање на Јакобиеви полиноми во ред од Бернулиеви полиноми.