A CLASS OF POLYNOMIALS CONNECTED WITH LAGUERRE POLYNOMIALS

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We construct a class of polynomials belonging to the vector space V of polynomials of degree less or equal to n whose basis are the Laguerre polynomials $L_n^{(\alpha)}(x)$. They have a given common zero $a \in R$ *or* C. The other zeros are in the interval $(0,\infty)$, which contains the zeros of $L_n^{(\alpha)}(x)$.

1. Let $L_n^{(\alpha)}(x)$, $n=0,1,2,...$ be the set of Laguerre polynomials defined by the relations

$$nL_n^{(\alpha)}(x) = (2n-1+\alpha-x)L_{n-1}^{(\alpha)}(x) - (n-\alpha+1)L_{n-2}^{(\alpha)}(x),$$
$$L_0^{(\alpha)}(x) = 1, L_{-1}^{(\alpha)}(x) = 0.$$

Consider the polynomial $\Lambda_n^{(\alpha)}(x)$ of degree $n(n \geq 2)$, given by

$$\Lambda_n^{(\alpha)}(x) = \begin{vmatrix} n-1-x+a & 1-n & 0 & \dots & 0 \\ 1-n-\alpha+x & 2n-3+\alpha-x & 2-n & \dots & 0 \\ a & 2-n+\alpha & 2n-5+\alpha-x & \dots & 0 \\ a & 0 & 3-n+\alpha & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ .. & \dots & \dots & \dots & \dots \\ a & 0 & \dots & \dots & 1-\alpha+x \end{vmatrix}$$

If we add to the elements of the first column the elements of the other columns until the n-th, we note that the elements of the first column of transformed determinant have the factor $x-a$ in common, which leads to the relation

$$\Lambda_n^{(\alpha)}(x) = \frac{a-x}{n}\left(L_{n-1}^{(\alpha)}(x) + L_{n-2}^{(\alpha)}(x) + \dots + L_0^{(\alpha)}(x)\right) \tag{1}$$

Expanding the determinant, we find

$$\Lambda_n^{(\alpha)}(x) = L_n^{(\alpha)}(x) + \left(\frac{a-\alpha}{n} - 1\right)L_{n-1}^{(\alpha)}(x) + \\ + \frac{a}{n}\left(L_{n-2}^{(\alpha)}(x) + L_{n-3}^{(\alpha)}(x) + \dots + L_0^{(\alpha)}(x)\right) \tag{2}$$

By comparing (1) and (2) we obtain

$$L_n^{(\alpha)}(x) + \left(\frac{a-\alpha}{n} - 1\right)L_{n-1}^{(\alpha)}(x) + \frac{a}{n}\left(L_{n-2}^{(\alpha)}(x) + L_{n-3}^{(\alpha)}(x) + \dots + L_0^{(\alpha)}(x)\right) \\ = \frac{a-x}{n}\left(L_{n-1}^{(\alpha)}(x) + L_{n-2}^{(\alpha)}(x) + \dots + L_0^{(\alpha)}(x)\right) \tag{3}$$

We see that the zeros of the polynomial $\Lambda_n^{(\alpha)}(x)$, distinct from a, are the same as the zeros of the polynomial $\left(L_{n-1}^{(\alpha)}(x) + L_{n-2}^{(\alpha)}(x) + \dots + L_0^{(\alpha)}(x)\right)$. From (3) we have

$$L_n^{(\alpha)}(x) - \frac{\alpha+n}{n}L_{n-1}^{(\alpha)} = -\frac{x}{a}\left(L_{n-1}^{(\alpha)}(x) + \dots + L_0^{(\alpha)}(x)\right) \tag{4}$$

The zeros of $L_n^{(\alpha)}(x) - \frac{\alpha+n}{n}L_{n-1}^{(\alpha)}$ are real and belong to the interval $(0,\infty)$. Consequently, the zeros of the polynomial $\Lambda_n^{(\alpha)}(x)$, distinct from a, are in the same interval.

2. The relations (1) and (4) lead to the expression

$$\Lambda_n^{(\alpha)}(x)=\frac{x-a}{x}\left(L_n^{(\alpha)}(x)-\frac{\alpha+n}{n}L_{n-1}^{(\alpha)}(x)\right)$$

Taking into consideration that $L_n^{(\alpha+1)}=\sum_{k=0}^{n}L_k^{(\alpha)}(x)$

We obtain $\Lambda_n^{(\alpha)}(x)=\frac{a-x}{n}L_{n-1}^{(\alpha)}(x)$

More generally we find

$$\Lambda_{n,k}^{(\alpha)}(x)=L_n^{(\alpha)}(x)+\frac{a-\alpha}{n}L_{n-1}^{(\alpha)}(x)+\left(\frac{a}{n}+\frac{a-\alpha}{n-1}\right)L_{n-2}^{(\alpha)}(x)+\ldots+$$

$$+\left(\frac{a}{n}+\frac{a-\alpha}{n-1}+\ldots+\frac{a-\alpha}{n-k+1}-1\right)L_{n-k}^{(\alpha)}(x)$$

Where

$$\Lambda_{n,k}^{(\alpha)}(x)=\Lambda_n^{(\alpha)}(x)+\Lambda_{n-1}^{(\alpha)}(x)+\ldots+\Lambda_{n-k}^{(\alpha)}(x)$$

We have also

$$\Lambda_{n,k}^{(\alpha)}(x)=\frac{x-a}{x}\left(L_n^{(\alpha)}(x)-\frac{\alpha}{n}L_{n-1}^{(\alpha)}(x)-\frac{\alpha}{n-1}L_{n-2}^{(\alpha)}(x)+\right.$$

$$\left.-\ldots-\left(\frac{\alpha}{n-k-1}+1\right)L_{n-k-1}^{(\alpha)}(x)\right)$$

and

$$\Lambda_{n,k}^{(\alpha)}(x)=(a-x)\left(\frac{1}{n}L_{n-1}^{(\alpha+1)}(x)+\frac{1}{n-1}L_{n-2}^{(\alpha+1)}(x)+\ldots+\frac{1}{n-k}L_{n-k-1}^{(\alpha+1)}(x)\right)$$

3. Using the relation given above, we obtain some definite integrals. We find

$$1^\circ\quad \int_0^\infty e^{-x}x^{m+\alpha}\Lambda_n^{(\alpha)}(x)dx=\begin{cases}(-1)^n\binom{m}{n}\Gamma(m+\alpha+1)\left(\frac{m+\alpha+1}{m-n+1}-\frac{a}{m}\right), m\geq n-1\\ 0, m<n+1\end{cases}$$

and

$$2^\circ\quad \int_0^\infty e^{-x}x^{k+1}\left(\Lambda_n^{(\alpha)}(x)\right)^2dx$$

$$=\frac{(m+n)!}{n!n}\left(a^2-2a(2n+k)+6n(n+k)+k(k+1)\right), k\in N$$

and more generally

$$3^\circ\quad \int_0^\infty e^{-x}x^{m+\alpha}\Lambda_{n,k}^{(\alpha)}(x)dx=\sum_{r=0}^{k}(-1)^{m-n}\binom{m}{n-r}\left(\frac{m+\alpha+1}{m-n-r+1}-\frac{a}{m}\right), m<n-k.$$

References

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