

**SOME RELATIONS BETWEEN BESSEL
AND BERNOULLI POLYNOMIALS**

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An expansion of Bessel polynomials through Bernoulli polynomials is given.

1. The simple Bessel polynomials

$$y_n(x) = {}_2F_0\left(-n, 1+n; -; -\frac{x}{2}\right)$$

and the generalized one

$$y_n(a, b; x) = {}_2F_0\left(-n, a-1+n; -; -\frac{x}{b}\right)$$

may be considered as special cases of the polynomials

$$\varphi_n(c; x) = \frac{(c)_n}{n!} {}_2F_0(-n, c+n; -; x)$$

Really, we have $\varphi_n\left(1, -\frac{x}{2}\right) = y_n(x)$

and $\frac{n!}{(a-1)_n} \varphi_n\left(a-1, -\frac{x}{b}\right) = y_n(a, b; x)$

respectively [1].

${}_2F_1(\alpha, \beta; \gamma; x)$ is hypergeometric function defined by

$${}_2F_1(\alpha, \beta; \gamma; x) = \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n x^n}{(\gamma)_n n!}$$

with $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1), n \geq 1; (\alpha)_0 = 1, \alpha \neq 0$

the Pochhammer symbol. Similarly, ${}_pF_q(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x)$ is generalized hypergeometric function used later.

The Bernoulli polynomials $B_n(x)$ are defined by [2]

$$\frac{te^{tx}}{e^t - 1} = \sum_{k=0}^n B_k(x) \frac{t^k}{k!} \tag{1}$$

Two of the properties of Bernoulli polynomial arising from (1) are

$$B_n(x+1) - B_n(x) = nx^{n-1}, n = 0, 1, 2, \dots \tag{2}$$

$$B_n(x+1) = \sum_{k=0}^n \binom{n}{k} B_k(x) \tag{3}$$

From (2) and (3) we find

$$x^n = \sum_{k=0}^n \frac{n! B_k(x)}{k!(n-k+1)!} \tag{4}$$

the expansion (4) is used bellow in the method of technique applied for expanding the polynomials.

The purpose of this paper is to give the expansions of the polynomial $\varphi_n(x, c)$ and the Bessel polynomials $y_n(x)$ and $y_n(a, b, x)$ in series of Bernoulli polynomials.

2. Consider the series

$$\sum_{k=0}^{\infty} \varphi_n(c, x) t^k = \sum_{k=0}^{\infty} \sum_{s=0}^n \frac{(-1)^s (c)_{n+s} x^s t^{n+s}}{s!(n-s)!} = \sum_{n, s=0}^{\infty} \frac{(-1)^s (c)_{n+2s} x^s t^{n+s}}{s! n!}$$

$$\begin{aligned}
&= \sum_{n,s=0}^{\infty} \sum_{k=0}^s \frac{(-1)^s (c)_{n+2s} B_k(x) t^{n+s}}{n!(s-k+1)!} \\
&= \sum_{n,k,s=0}^{\infty} \frac{(-1)^{s+k} (c)_{n+2s+2k} B_k(x) t^{n+s+k}}{n!k!(s+1)!}
\end{aligned}$$

in which we have used the identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^n A(n,k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,n+k) \tag{5}$$

to collect the powers of t in the last summation above.

Using the same identity (5) in reverse, we may write

$$\begin{aligned}
\sum_{n=0}^{\infty} \Phi_n(c,x) t^n &= \sum_{n,k=0}^{\infty} \sum_{s=0}^n \frac{(-1)^{s+k} (c)_{n+s+2k} B_k(x) t^{n+k}}{(s+1)!(n-s)!k!} \\
&= \sum_{n,k=0}^{\infty} \sum_{s=0}^n \frac{(-1)^k (c+n+2k)_s (-n)_s (1)_s (c)_{n+2k} B_k(x) t^{n+k}}{s!(2)_s n!k!} \\
&= \sum_{n,k=0}^{\infty} {}_3F_1(-n, c+n+2k, 1; 2; 1) \frac{(-1)^k (c)_{n+2k} B_k(x) t^{n+k}}{n!k!}
\end{aligned}$$

Applying the identity (5) again, we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} \Phi_n(c,x) t^n &= \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^n {}_3F_1(-n+k, c+n+k, 1; 2; 1) \frac{(-1)^k (c)_{n+k} B_k(x) t^n}{k!(n-k)!}
\end{aligned}$$

Finally we have

$$\Phi_n(c,x) = \sum_{k=0}^n {}_3F_1(-n+k, c+n+k, 1; 2; 1) \frac{(-1)^k (c)_{n+k} B_k(x)}{k!(n-k)!}$$

3. For the Bessel polynomials we find

$$y_n(x) = \sum_{k=0}^n {}_3F_1(-n+k, 1+n+k, 1; 2; 1) \frac{(-1)^k (n+k)! B_k\left(-\frac{k}{2}\right)}{k!(n-k)!} \text{ or}$$

$$y_n(x) = \sum_{k=0}^n {}_3F_1\left(-n+k, 1+n+k, 1; 2; -\frac{1}{2}\right) \frac{(n+k)! B_k(x)}{k!(n-k)! 2^k}$$

and

$$\begin{aligned}
y_n(a,b;x) &= \\
&= \sum_{k=0}^n {}_3F_1(-n+k, a-1+k+n, 1; 2; 1) (-1)^k \binom{n}{k} (a-1+n)_k B_k\left(-\frac{x}{b}\right)
\end{aligned}$$

or

$$\begin{aligned}
y_n(a,b;x) &= \\
&= \sum_{k=0}^n {}_3F_1\left(-n+k, a-1+k+n, 1; 2; -\frac{1}{b}\right) \binom{n}{k} \frac{(a-1+n)_k}{b^k} B_k(x)
\end{aligned}$$

References

1. E. Rainville, Special functions, The Macmillan Company, New York, 1960
2. A. Erdelyi et al. Higher Transcendental Functions, Vol. 1, McGraw-Hill, New York-Toronto-London, 1952