EXPRESSIONS OF LAGUERRE POLYNOMIALS
THROUGH BERNOULLI POLYNOMIALS
Математички Билтен, 22 (XLVIII), 1998, 15-18

Abstract

The aim of this paper is to obtain an expansion formula of Larguerre polynomials through Bernoulli polynomials.

1. Introduction

Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined by

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_1F_1\,(-n;\,1+\alpha;\,x), \qquad n \geq 0 \tag{1}$$

where

$${}_1F_1\,(a;\,b;\,x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n\, n!}$$

is the confluent hypergeometric function and

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\,,\ \ n>1; \quad (\alpha)_0 = 1\,,\ \alpha \neq 0$$

is the factorial function [1]. It is a special case of the generalized hypergeometric function ${}_pF_q\,(\alpha_1,\,\alpha_2,\,\dots,\,\alpha_p;\ \beta_1,\,\beta_2,\,\dots,\,\beta_q;\,x)$ used later.

From (1) it follows immediately that

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k(1+\alpha)_n x^k}{k!(n-k)!\,(1+\alpha)_k}\,.$$

Bernoulli polynomials $B_n(x)$ are defined by

$$\frac{te^{tx}}{e^t-1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!} \tag{2}$$

from where we have

$$B_n(x) = \sum_{r=0}^{n} \binom{n}{r} B_r x^{n-r}\,. \tag{3}$$

B_r are the Bernoulli nombers [2].

From (2) it follows

$$\sum_{n=0}^{\infty} (B_n(x+1) - B_n(x))\,\frac{t^n}{n!} = te^{tx} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!}\,t^n$$

and we have

$$B_n(x+1) - B_n(x) = nx^{n-1}\,, \qquad n = 0,\,1,\,2,\,\dots\,. \tag{4}$$

We use the relations (3) and (4) to conclude that

$$x^n = \sum_{k=0}^{n} \frac{n!\,B_k(x)}{k!\,(n-k+1)!}\,. \tag{5}$$

2. The relation between Laguerre and Bernoulli polynomials

Consider the series

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s(1+\alpha)_n x^s t^n}{s!\,(n-s)!\,(1+\alpha)_s}$$

$$= \sum_{n,\,s=0}^{\infty} \frac{(-1)^s(1+\alpha)_{n+s}\,x^s t^{n+s}}{s!\ n!\ (1+\alpha)_s}\,.$$

Using (5) we have

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n,\, s=0}^{\infty} \sum_{k=0}^{s} \frac{(-1)^s(1+\alpha)_{n+s}B_k(x)t^{n+s}}{n!\; k!\; (s-k+1)!\; (1+\alpha)_s} =$$
$$= \sum_{n,\, k,\, s=0}^{\infty} \frac{(-1)^{s+k}(1+\alpha)_{n+s+k}B_k(x)t^{n+k+s}}{n!\; k!\; (s+1)!\; (1+\alpha)_{s+k}}$$

in which we use the identity

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} A(k,\, n-k) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} A(k,\, n) \tag{6}$$

to collect the powers of t in the last summation above. By the same identity used conversely we may write

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n,k=0}^{\infty} \sum_{s=0}^{n} \frac{(-1)^{s+k}(1+\alpha)_{n+k}B_k(x)t^{n+k}}{(n-s)!(1+\alpha)_{s+k}k!(s+1)!} =$$
$$= \sum_{n,k=0}^{\infty} {}_2F_2(-n, 1; 1+\alpha+k, 2; 1)\frac{(-1)^k(1+\alpha)_{n+k}B_k(x)t^{n+k}}{n!k!(1+\alpha)_k}.$$

By the identity (6) used again conversely, we obtain

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty} \sum_{k=0}^{n} {}_2F_2(-n+k, 1; 1+\alpha+k, 2; 1)\frac{(-1)^k(1+\alpha)_n B_k(x)t^n}{k!(n-k)!(1+\alpha)_k}.$$

Finally, it is

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_2F_2\; (-n+k.\, 1;\; 1+\alpha+k,\; 2;\; 1)\frac{(-1)^k(1+\alpha)_n B_k(x)}{k!\,(n-k)!\,(1+\alpha)_k}\,. \tag{7}$$

3. Special cases

1° When $\alpha = 0$, we have the simple Laguerre polynomials denoted by $L_n(x)$ i.e.

$$L_n(x) = L_n^{(0)}(x) = {}_1F_1\,(-n;\; 1;\; x)\,.$$

Then from (7) we obtain

$$L_n(x) = \sum_{k=0}^{n} {}_2F_2(-n+k,\; 1;\; k+1,\; 2;\; 1)\, \frac{(-1)^k n! B_k(x)}{k!\, k!\, (n-k)!}\,.$$

2° Hermite polynomials reduced to Laguerre polynomials [3] give with (7) the relations

$$H_{2n}(x) = \sum_{k=0}^{n} {}_2F_2(-n+k,\; 1;\; k+\frac{1}{2},\; 2;\; 1)\binom{n}{k}\frac{(-1)^{n+k}2^{n+k}(2n-1)!!\; B_k(x^2)}{(2k-1)!!}$$

and

$$H_{2n+1}(x) = x\sum_{k=0}^{n} {}_2F_2(-n+k, 1; k+\frac{3}{2}, 2; 1)\binom{n}{k}\frac{(-1)^{n+k}2^{n+k+1}(2n+1)!! B_k(x^2)}{(2k+1)!!}.$$

References

[1] A. Erdelyi et al.: *Higher Transcedental Functions*, vol. 1, Mc-Graw-Hill, New York-Toronto-London, 1952.

[2] E.D. Rainville: *Special Functions*, The Macmillan Company, New York. 1960.

[3] G. Szegö: *Orthogonal Polynomials*, American Mathematical Society, New York, 1950.

ИЗРАЗУВАЊЕ НА ПОЛИНОМИТЕ НА LAGUERRE СО ПОЛИНОМИТЕ НА BERNOULLI

Резиме

Во трудот се дава едно развивање на полиномите на Laguerre во редови на полиномите на Bernoulli.