# EXPANSION OF LAGUERRE POLYNOMIALS INTO SERIES OF BESSEL POLYNOMIALS

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A b s t r a c t : Some relations between Laguerre polynomials and Bessel polynomials are given.

**1.** In the study of orthogonal polynomials one uses very often the hypergeometryc function ${}_2F_1(a, b; c; x)$ defined by [1]

$$ {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!} $$

with

$$ (\alpha)_n = \alpha(\alpha+1)(\alpha+2)...(\alpha+n-1), \qquad n \geq 1 $$

$$ (\alpha)_0 = 1, \ \ \alpha \neq 0 $$

the Pochammer symbol. Similarly ${}_pF_q(\alpha_1, \alpha_2, ..., \alpha_p; \beta_1 \beta_2, ..., \beta_q; x)$ is a generalised hypergeometric function.

The Laguerre polynomials $L_n^{(\alpha)}(x)$ are defined [1] for a $n$ nonnegative integer by

$$ L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} \ {}_1F_1(-n; 1+\alpha; x) $$

$$ = \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n x^k}{k!(n-k)!(1+\alpha)_k}. $$

This equation yields

$$ {}_0F_1(-; 1+\alpha; -xt) = e^{-t} \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x) t^n}{(1+\alpha)_n}. $$

Therefore

$$ \sum_{n=0}^{\infty} \frac{(-x)^n t^n}{(1+\alpha)_n n!} = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k} L_n^{(\alpha)}(x) t^n}{(n-k)!(1+\alpha)_k} $$

from which it follows that

$$ x^n = \sum_{k=0}^{n} \frac{(-1)^k n!(1+\alpha)_n L_k^{(\alpha)}(x)}{(n-k)!(1+\alpha)_k}. $$

The simple Bessel polynomials

$$ y_n(x) = {}_2F_0(-n, 1+n; -; -\frac{x}{2}) $$

and the generalised one

$$ y_n(a, b, x) = {}_2F_0(-n, a-1+n; -; -\frac{x}{b}) $$

may be considered as special cases of the polynomials

$$ \varphi_n(c, x) = \frac{(c)_n}{n!} \ {}_2F_0(-n, c+n; -; x). \qquad (1) $$

Really, we have

$$\varphi_n(1, -\frac{x}{2}) = y_n(x)$$

and

$$\frac{n!}{(a-1)_n}\varphi_n(a-1, -\frac{x}{b}) = y_n(a, b, x)$$

respectively.

The equation (1) yields the property

$$x^n = n! \sum_{k=0}^{n} \frac{(-1)^k (c+2k)\varphi_k(c, x)}{(n-k)!(c)_{n+k+1}}. \tag{2}$$

The aim of this paper is to give some relations between the Laguerre polynomials and the Bessel polynomials, using a classical method [1].

**2.** Consider the series

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s (1+\alpha)_n x^s t^n}{s!(n-s)!(1+\alpha)_s}$$

$$= \sum_{n,s=0}^{\infty} \frac{(-1)^s (1+\alpha)_{n+s} x^s t^{n+s}}{s!n!(1+\alpha)_s}.$$

Using (2) we may write

$$\sum_{k=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n,\, s=0}^{\infty}\sum_{k=0}^{s} \frac{(-1)^{s+k}(1+\alpha)_{n+s}(c+2k)\varphi_k(c,x)t^{n+s}}{n!(s-k)!(c)_{s+k+1}(1+\alpha)_s}$$

$$= \sum_{n,k,s=0}^{\infty} \frac{(-1)^s (1+\alpha)_{n+s+k}(c+2k)\varphi_k(c,x)t^{n+s+k}}{n!s!(c)_{s+2k+1}(1+\alpha)_{s+k}} \tag{3}$$

in which we have used the identity [1]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k, n-k),$$

to collect the powers of $t$ in the last summation above.

By the same identity used in reverse, from (3) we obtain

$$\sum_{n=0}^{\infty} L_n^{(\alpha)}(x)t^n = \sum_{n,k=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s (1+\alpha)_{n+k}(c+2k)\,\varphi_k(c,x)\,t^{n+k}}{(n-s)!s!(c)_{s+2k+1}(1+\alpha)_{s+k}}$$

$$= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n} \frac{(-n)_s (1+\alpha)_{n+k}(c+2k)\,\varphi_k(c,x)\,t^{n+k}}{n!\,s!(c+2k+1)_s(c)_{2k+1}(1+\alpha+k)_s(1+\alpha)_k}$$

$$= \sum_{n,k=0}^{\infty} {}_1F_2(-n; 1+c+2k, 1+\alpha+k; 1)\frac{(c+2k)(1+\alpha)_{n+k}\ \varphi_k(c,x)\,t^{n+k}}{n!(c)_{2k+1}(1+\alpha)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_1F_2(-n+k; 1+c+2k, 1+\alpha+k; 1)\frac{(1+\alpha)_n\varphi_k(c,x)\,t^n}{(n-k)!(c)_{2k}(1+\alpha)_k}.$$

Therefore

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_1F_2(-n+k; 1+c+2k, 1+\alpha+k; 1)\frac{(1+\alpha)_n \varphi_k(c,x)}{(n-k)!(c)_{2k}(1+\alpha)_k}.$$

For the simple Bessel polynomials we find

$$L_n^{(\alpha)}(-\frac{x}{2}) = \sum_{k=0}^{n} {}_1F_2(-n+k; 2+2k, 1+\alpha+k; 1)\frac{(1+\alpha)_n y_n(x)}{(n-k)!(2k)!(1+\alpha)_k}$$

or

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_1F_2(-n+k; 2k+2, 1+\alpha+k; -2)\frac{(-2)^k(1+\alpha)_n y_n(x)}{(n-k)!(2k)!(1+\alpha)_k}$$

and for the generalised Bessel polynomials we find

$$L_n^{(\alpha)}(-\frac{x}{b}) = \sum_{k=0}^{n} {}_1F_2(-n+k; a+2k, 1+\alpha+k; 1)\frac{(1+\alpha)_n y_n(a,b,x)}{k!(n-k)!(a+k-1)_k(1+\alpha)_k}$$

or

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_1F_2(-n+k; a+2k, 1+\alpha+k; -b)\frac{(-b)^k(1+\alpha)_n y_n(a,b,x)}{(n-k)!(a-1)_{2k}(1+\alpha)_k}.$$

**3.** Following the same procedure, let us expand the Bessel polynomial, in a series of Laguerre polynomials. Consider the series

$$\begin{aligned}\sum_{n=0}^{\infty} \varphi_n(c,x)t^n &= \sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(c)_{s+n}x^s t^n}{s!(n-s)!}\\ &= \sum_{n,s=0}^{\infty}\frac{(-1)^s(c)_{n+2s}x^s t^{n+s}}{s!n!}\\ &= \sum_{n,s=0}^{\infty}\sum_{k=0}^{s}\frac{(-1)^{s+k}(c)_{n+2s}(1+\alpha)_s L_k^{(\alpha)}(x)t^{n+s}}{n!(s-k)!(1+\alpha)_k}\\ &= \sum_{n,k,s=0}^{\infty}\frac{(-1)^s(c)_{n+2s+2k}(1+\alpha)_{s+k}L_k^{(\alpha)}(x)t^{n+s+k}}{n!s!(1+\alpha)_k}.\end{aligned}$$

Now if we rearrange terms in the above, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \varphi_n(c,x)t^n &= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n}\frac{(-n)_s(c+n+2k)_s(1+\alpha+k)_s(c)_{n+2k}(1+\alpha)_k L_n^{(\alpha)}(x)t^{n+s}}{n!s!(1+\alpha)_k}\\ &= \sum_{n,k=0}^{\infty} {}_3F_0(-n, c+n+2k, 1+\alpha+k; -; 1)\frac{(c)_{n+2k}L_k^{(\alpha)}(x)t^{n+k}}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_3F_0(-n+k, c+n+k, 1+\alpha+k; -; 1)\frac{(c)_{n+k}L_k^{(\alpha)}(x)t^n}{(n-k)!}.\end{aligned}$$

Finally

$$\varphi_n(c,x) = \sum_{k=0}^{n} {}_3F_0(-n+k, 1+n+k, 1+\alpha+k; -; 1)\frac{(c)_{n+k}L_k^{(\alpha)}(x)}{(n-k)!}.$$

For the simple Bessel polynomials we find

$$y_n(x) = \sum_{k=0}^{n} {}_3F_0(-n+k, 1+n+k, 1+\alpha+k; -; 1) \frac{(n+k)!\, L_k^{(\alpha)}(-\frac{x}{2})}{(n-k)!}$$

or

$$y_k(x) = \sum_{k=0}^{n} {}_3F_0(-n+k, n+k+1, 1+\alpha+k; -; -\frac{1}{2}) \frac{(-1)^k (n+k)!\, L_k^{(\alpha)}(x)}{2^k (n-k)!}$$

and for the generalised Bessel polynomials we have

$$y_n(a,b,x) = \sum_{k=0}^{n} {}_3F_0(-n+k, a+n+k-1, 1+\alpha+k; -; 1) \frac{n!(a+n-1)_k\, L_k^{(\alpha)}(-\frac{x}{b})}{(n-k)!}$$

or

$$y_n(a,b,x) = \sum_{k=0}^{n} {}_3F_0(-n+k, a+n+k-1, 1+\alpha+k; -; -\frac{1}{b}) \frac{(-1)^k n!(a+n-1)_k\, L_k^{(\alpha)}(x)}{b^k (n-k)!}$$

**4.** *Special cases*. For the simple Laguerre polynomials $L_k^{(0)}(x) = L_k(x)$ arising frequently, considered as special cases, we give

1° $\varphi_n(c,x) = \sum_{k=0}^{n} {}_3F_0(-n+k, n+k+1, k+1; -; 1) \frac{(c)_{n+k} L_k(x)}{(n-k)!}$

2° $y_n(x) = \sum_{k=0}^{n} {}_3F_0(-n+k, n+k+1, k+1; -; -\frac{1}{2}) \frac{(-1)^k (n+k)!\, L_k(x)}{2^k (n-k)!}$

3° $y_n(a,b,x) = \sum_{k=0}^{n} {}_3F_0(-n+k, a+k+n-1, k+1; -; -\frac{1}{b}) \frac{(-1)^k n!(a+n-1)_k L_k(x)}{b^k (n-k)!}$

4° $L_n(x) = \sum_{k=0}^{n} {}_1F_2(-n+k; c+2k+1, k+1; -; -1) \binom{n}{k} \frac{\varphi_k(c,x)}{(c)_{2k}}$

5° $L_n(x) = \sum_{k=0}^{n} {}_1F_2(-n+k; 2k+2, k+1; -2)(-2)^k \binom{n}{k} \frac{y_k(x)}{(2k)!}$

6° $L_n(x) = \sum_{k=0}^{n} {}_1F_2(-n+k; 2k+2, k+1; -b)(-b)^k \binom{n}{k} \frac{y_n(a,b,x)}{(a-1)_{2k}}$.

REFERENCES


[1] Rainville E. D., *Special Functions*, The Macmillan Company, New York (1960).

[2] Szegö, G., *Orthogonal Polynomials*, American Mathematical Society, New York (1950).



Резиме

**РАЗВИВАЊЕ НА ПОЛИНОМИ НА LAGUERRE ВО РЕДОВИ ОД ПОЛИНОМИ НА BESSEL**

Во трудот се даваат некои релации помеѓу полиноми на Laguerre и полиноми на Bessel.