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ON ALGEBRAIC STRUCTURES FOR MATRICES, RELATIONS AND GRAPHS OVER A SEMIRING

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Abstract. Matrices, relations and graphs over a semiring are considered. The algebraic operations and the corresponding algebraic structures as well as their relationship are studied.

1. Motivation

Each graph defines a binary relation over a finite or infinite set and vice versa [1], [2], [8], [11]. On the other side we can assign to each graph a matrix with elements $m_{ij}=1$ if there exists an edge from the vertex v_i to the vertex v_j and $m_{ij}=0$ otherwise. These connections between graphs, relations and matrices over the Boolean semiring $\mathbb{B}$ are completely studied in [1], [8], [11].

Naturally there arise the following problems for the general case:

i) how to define graphs, relations and matrices over a semiring;

ii) hov to define the algebraic operations (generalizing the usual) with such matrices, relations and graphs;

iii) is there any connection between graphs, relations and matrices over a semiring;

iv) give a certain interpretation of these results in graph theory and its applications.

The above marked problems are object of this paper. An extended summary of the paper is given in [13].

2. Preliminaries

We recall the definitions of semiring [4] and semimodule [11]. The terminology and the notations not especially indicated in the paper are according to [9], [10] for the category theory and algebra respectively.

A semiring is an algebra $\mathbb{K}=(K,+,.,0,1)$, where:

i) $(K,+,0)$ is a commutative monoid with 0 as neutral element;

ii) $(K,.,1)$ is a monoid with neutral element 1;

iii) the operation . is distributive over the operation +, i.e.

$$a\cdot(b+c)=a\cdot b+a\cdot c \text{ and } (a+b)\cdot c=a\cdot c+b\cdot c \text{ for each } a,b,c\in K;$$

iv) $a\cdot 0=0\cdot a=0$ for each $a\in K$.

Onviously each ring is a semiring and each bounded chain too. We list a number of semirings that are not rings and that will be of interest to the next exposition.

Examples. 1°. $\mathbb{B}=(\{0,1\},\vee,\wedge,0,1)$ is the Boolean semiring [4] with operations $\vee$ (disjunction) and $\wedge$ (conjunction) and with neutral elements respectively 0 and 1;

2°. $\mathbb{F}=([0,1],\max,\min,0,1)$ - the bounded chain over the interval $[0,1]\subset\mathbb{R}$ with operations max=sup and min=inf and according to the natural order in $\mathbb{R}$ [12]. This semiring is fundamental for the fuzzy set theory [3], [8], [14], [16];

3°. $\mathbb{L}=(L,\vee,\wedge,0,1)$ - the bounded chain over the ordered set L with lower and upper bounds 0 and 1 respectively. This semiring is a natural generalization of the semiring $\mathbb{F}$ [5];

4°. $\mathbb{P}(Y)=(2^{Y^*},\cup,.,\emptyset,\{\Lambda\})$ [1] is the strings semiring with $\cup$ (union) with unit $\emptyset$ and . (concatenation) with unit $\{\Lambda\}$.

5°. $\mathbb{N}$ - the semiring of all integers $n\geq 0$ with the usual addition and multiplication [1], [4].

6°. $\mathbb{R}_+$ [4], [11] - the semiring of all nonnegative real numbers with the usual addition and multiplication.

A semimodule over the semiring $\mathbb{K}$ is the algebra $\mathbb{H}=(H,\mathbb{K},+,.,0,1)$ where H is a set, $+: H\times H\to H$ and $.: K\times H\to H$ are operations and:

i) $(H,+,0)$ is a commutative monoid with 0 as neutral element;

ii) The two structures are connected with the following axioms:

$$\alpha\cdot(\beta\cdot a)=(\alpha\cdot\beta)\cdot a;\quad (\alpha+\beta)\cdot a=\alpha\cdot a+\beta\cdot a;\quad \alpha\cdot(a+b)=\alpha\cdot a+\beta\cdot b;$$

$$1\cdot a=a\cdot 1=a;\quad 0\cdot a=a\cdot 0=0$$

for arbitrary $\alpha,\beta\in K$, $a,b\in H$ (here 0 and 1 are the neutral elements in $\mathbb{K}$).

Clearly any module is a semimodule. Other examples are given in the next text.

3. Matrices, relations and graphs over a semiring

In what follows we write $\mathbb{K}$ for the semiring $\mathbb{K}=(K,+,.,0,1)$.

Let I,J be sets. The matrix $M_{IxJ}=(m_{ij})$ with elements $m_{ij}\in K$ for each $(i,j)\in IxJ$ is a matrix over the semiring $\mathbb{K}$. Formally the index sets I,J may be finite or infinite, but the infinite sets do not have sense for the practice. We define the following algebraic operations with matrices over $\mathbb{K}$, using the semiring operations [1]: The matrix $M_{IxJ}=(m_{ij})$ is the sum for $M'_{IxJ}=(m'_{ij})$ and $M''_{IxJ}=(m''_{ij})$ if

$$m_{ij} = m'_{ij} + m''_{ij} \tag{1}$$

The matrix $0_{IxJ}=(0)$ for each $(i,j)\in IxJ$ is the neutral element for each M_{IxJ} with respect to the addition. The matrix $M_{IxJ}=(m_{ij})$ is the product for $M'_{IxK}=(m'_{ik})$ and $M''_{KxJ}=(m''_{kj})$ if

$$m_{ij} = \sum_{k\in K} m'_{ik}\cdot m''_{kj} \tag{2}$$

The square identity matrices E_{IxI} and E_{JxJ} with $e_{kp}=1$ if k=p and $e_{kp}=0$ otherwise are respectively left and right unit for each M_{IxJ} with respect to multiplication. For $M_{IxJ}=(m_{ij})$ and $\alpha\in K$ we define the scalar multiplication by the equation

$$\alpha\cdot M_{IxJ} = (\alpha\cdot m_{ij}) \tag{3}$$

Examples. 1^o. For $\mathbb{K}=\mathbb{N}$, $\mathbb{R}_+$ we have the usual addition, scalar multiplication and multiplication for matrices.

2^o. For $\mathbb{K}=\mathbb{B}$, $\mathbb{F}$, $\mathbb{L}$ we have $m'_{ij} + m''_{ij} = m'_{ij}\vee m''_{ij}$ and $m_{ij} = \sum_{k\in K} m'_{ik}\cdot m''_{kj} = \bigvee_{k\in K}(m'_{ik}\wedge m''_{kj})$, for instance

$$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}+\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix}=\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix};\quad \begin{pmatrix}0,3 & 0,1\\0,2 & 0,4\end{pmatrix}\cdot\begin{pmatrix}0,1 & 0,2\\0,5 & 0\end{pmatrix}=\begin{pmatrix}0,1 & 0,2\\0,4 & 0,2\end{pmatrix}$$

3^{o}. For $\mathbb{K}=\mathbb{P}(Y)$ for instance we obtain

$$\begin{pmatrix}\{y_1\} & \{y_1,y_2\}\\ \{\Lambda\} & \{y_1\}\end{pmatrix}+\begin{pmatrix}\{y_2\} & \{y_1\cdot y_3\}\\ \{\Lambda\} & \{\Lambda\}\end{pmatrix}=\begin{pmatrix}\{y_1,y_2\} & \{y_1,y_2,y_1\cdot y_3\}\\ \{\Lambda\} & \{y_1\}\end{pmatrix}$$

$$\begin{pmatrix}\{y_1\} & \{y_1,y_2\}\\ \{\Lambda\} & \{y_1\}\end{pmatrix}\cdot\begin{pmatrix}\{y_2\} & \{y_1\cdot y_3\}\\ \{\Lambda\} & \{\Lambda\}\end{pmatrix}=\begin{pmatrix}\{y_1\cdot y_2,y_1,y_2\} & \{y_1\cdot y_1\cdot y_3,y_1,y_2\}\\ \{y_2,y_1\} & \{y_1\cdot y_3,y_1\}\end{pmatrix}$$

Let $M_{IxJ}(\mathbb{K})$ be the set of all IxJ-matrices over $\mathbb{K}$.

PROPOSITION 1. i) $(M_{IxJ}(\mathbb{K}),+,0_{IxJ})$ is a commutative monoid with neutral element the null-matrix 0_{IxJ};

ii) $(M_{IxJ}(\mathbb{K}),+,.,0_{IxJ},1)$ is a semimodule over $\mathbb{K}$;

iii) $(M_{IxI}(\mathbb{K}),.,E_{IxI})$ is a monoid with neutral element E_{IxI} [1];

iv) $(M_{IxI}(\mathbb{K}),+,.,0_{IxI},E_{IxI})$ is a semiring [1].

We shall define relation over a semiring and the related algebraic operations.

Let A,B be sets. $R=\{(a,b,k)\}\subset AxBxK$ is called a (binary) <u>relation over the semiring</u> $\mathbb{K}$. The relation R contains all pairs $(a,b)\in AxB$ with their scalar estimation $k\in K$.

Clearly this definition of a relation over a semiring includes as a partial case the classical definition. If we consider $\mathbb{K}=\mathbb{B}$ and a relation $R=\{(a,b,k)\}\subset AxBx\{0,1\}$, we have the well-known notation of relation [2], [10]. Usually such relations are described only by pairs $(a,b)\in AxB$ with k=1. For $\mathbb{K}=\mathbb{F}$ we obtain the fuzzy relations [3], [8].

Using the semiring operations we define the operations with relations. The relation

$$R=R_1+R_2=\{(a,b,k_1+k_2)\mid (a,b,k_1)\in R_1 \;\&\; (a,b,k_2)\in R_2\}\subset AxBxK \quad (4)$$

is the <u>sum</u> for the relations $R_1=\{(a,b,k_1)\}\subset AxBxK$ and $R_2=\{(a,b,k_2)\}\subset AxBxK$. The relation $R_0=\{(a,b,0)$ with k=0 for each $(a,b) \in A \times B\}$ is the neutral element for each $R \subset A \times B \times K$ with respect to the addition. <u>Scalar multiplication</u> for a relation $R=\{(a,b,k)\} \subset A \times B \times K$ and $\alpha \in K$ is the relation

$$\alpha \cdot R = \{(a,b,\alpha \cdot k) \mid (a,b,k) \in R\} \tag{5}$$

<u>Composition</u> for the relations $R_1 = \{(a,b,k_{ab})\} \subset A \times B \times K$ and $R_2 = \{(b,d,k_{bd})\} \subset B \times D \times K$ is the relation

$$R = R_1 o R_2 = \{(a,d,k) \mid (k = \sum_{b \in B} k_{ab} \cdot k_{bd})((a,b,k_{ab}) \in R_1 \ \& \ (b,d,k_{bd}) \in R_2)\} \subset A \times D \times K \tag{6}$$

The diagonal relations $\Delta = \{(z', z'', k_{z'z''}) \mid k_{z'z''}=1$ if $z'=z''$ and $k_{z'z''}=0$ if $z' \neq z''\} \subset Z \times Z \times K$ are respectively left ($Z=A$) and right ($Z=B$) unit for the relation $R \subset A \times B \times K$.

Clearly this definition for composition of relations includes the usual composition [2], [10] as a particular case. On the other side it shows how to generalize the composition of n-ary relations [15] over a semiring.

<u>Examples</u>. 1°. For $K=B$, F, L we have $R_1+R_2=R_1 \vee R_2 = \{(a,b,k_1 \vee k_2)\}$ for the addition; for the composition - the classical case for $K=B$ [2], [10]; the fuzzy relation composition [3], [8] for $K=F$, etc.

2°. For $K=P(Y)$ the addition is $R_1+R_2 = R_1 \cup R_2$; the composition is defined by the union and concatenation of the strings as follows: $R_1 o R_2 = \{(a,d,w_{ad}) \mid (w_{ad} = \bigcup_{b \in B} w_{ab} \cdot w_{bd})((a,b,w_{ab}) \in R_1 \ \& \ (b,d,w_{bd}) \in R_2)\}$. Here w_{ab}, w_{bd} and w_{ad} are strings from Y^*, $w_{ab} \cdot w_{bd}$ stands for the concatenation of the strings [1], [4].

Let $R_{A \times B}(K)$ denote the class of all small relations (i.e. A and B are sets).

PROPOSITION 2. i) $(R_{A \times B}(K),+,R_0)$ is a commutative monoid;

ii) $(R_{A \times B}(K),+,\cdot,R_0,1)$ is a semimodule over K;

iii) $(R_{A \times A}(K),o,\Delta)$ is a monoid;

iv) $(R_{A \times A}(K),+,o,R_0,\Delta)$ is a semiring.

The proof consists in direct verification of the axioms.

The algebra $G=(V,E,\alpha,\beta,\mu)$, where $V=A\times B$ and E are sets of nodes (or vertices) and edges respectively, $\alpha:E\to A$, $\beta:E\to B$, $\mu:E\to K$ are functions such that each $f\in E$ satisfies $\mathrm{dom}f=\alpha(f)$, $\mathrm{codf}=\beta(f)$ and $\mu(f)$ is the label (resp. the weight, membership degree, traffic capacity, etc.) is called a <u>directed graph over the semiring</u> $\mathbb{K}$.

Each directed graph over $\mathbb{K}$ can be visualized (fig. 1).

If the edges are not oriented the graph is undirected. In what follows we write graph instead of directed $\mathbb{K}$-graph over the semiring $\mathbb{K}$.

For each graph $G=(V,E,\alpha,\beta,\mu)$ the set of the edges E is isomorphic to the set of the triples $\{(a,b,k)\mid a=\alpha(f),\ b=\beta(f),\ k=\mu(f)$ for each $f\in E\}\subset A\times B\times K$. If $\alpha(f)=v_i$ and $\beta(f)=v_j$ then the edge f joins v_i to v_j with weight $k=\mu(f)$. If $k=0$ we often consider there is no edhe.

<u>Examples</u>. 1°. For $\mathbb{K}=\mathbb{B}$ we have the usual definition [11] for the directed graph (fig. 2). There is an edge f joining two vertices iff $\mu(f)\neq 0$.

2°. For $\mathbb{K}=\mathbb{F}$ the graph is fuzzy [8], cf. fig. 3; for $\mathbb{K}=\mathbb{L}$ we obtain the next hierarchical generalization, etc.

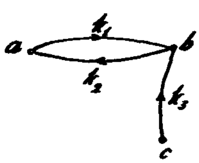

Fig. 1

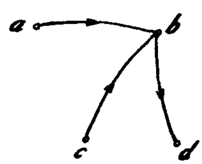

Fig. 2

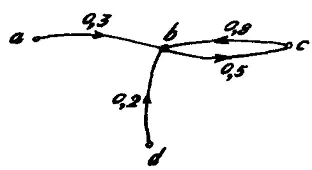

Fig. 3

3°. For $\mathbb{K}=\mathbb{P}(Y)$ the edges of the graph are labeled by the words of the free monoid Y^*, as it is very useful for the theory of abstract automata [2] cf. fig. 4.

4^{o}. For $K=R_+$ if the labels of the graph satisfy the condition $\sum_{q' \in B} \mu(f_q) = 1$, where $q' = \beta(f_q)$, for each $q \in A$, then the graph is stochastic (fig. 5).

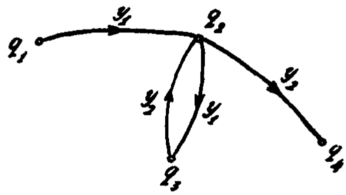

Fig. 4

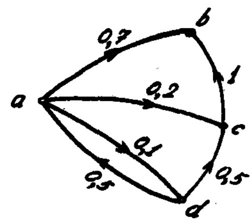

Fig. 5

Let $G(K)$ be the class of all small graphs [9]. According to the operations in K we define operations in $G(K)$.

For $G_1=(V,E_1,\alpha_1,\beta_1,\mu_1)$ and $G_2=(V,E_2,\alpha_2,\beta_2,\mu_2)$ the graph

$$G = G_1 + G_2 = (V,E,\alpha,\beta,\mu), \tag{7}$$

where $E=\{f/f=f_1+f_2, \alpha_1(f_1)=\alpha_2(f_2)=\alpha(f), \ \beta_1(f_1)=\beta_2(f_2)=\beta(f), \mu(f)=\mu_1(f_1)+\mu_2(f_2)\}$ is the sum of G_1 and G_2. The graph $G_0=(V,E,\alpha,\beta,0)$ is the neutral element for each $G=(V,E,\alpha,\beta,\mu)$ with respect to the addition. Here $0:\bar{E} \to K$ stands for the constant map, such that $0:f \to 0 \in K$ for each $f \in \bar{E}$, $\bar{\alpha}(\bar{E})=A$, $\bar{\beta}(\bar{E})=B$.

Let $G_1=(V_1,E_1,\alpha_1,\beta_1,\mu_1)$ and $G_2=(V_2,E_2,\alpha_2,\beta_2,\mu_2)$ with $B_1=A_2$ be two graphs. The graph $G=G_1oG_2$ is a composition for G_1 and G_2 if

$$\begin{aligned} V_{G_1oG_2} &= AxD, \text{ where } V_1=axB_1 \text{ and } V_2=A_2xD, \\ E_{G_1oG_2} &= \{f{\approx}(\alpha(f),\beta(f),\mu(f)/f = \sum_{b \in B_1} f_1 o f_2, \\ &\quad \alpha(f) = \alpha_1(f_1),\ \beta(f) = \beta_2(f_2),\ \alpha_2(f_2) = \beta_1(f_1), \\ &\quad f_1 \in E_1,\ f_2 \in E_2,\ \ \mu(f) = \sum_{b \in B_1} \mu_1(f_1)\cdot\mu_2(f_2)\} \end{aligned} \tag{8}$$

For $G=(V,E,\alpha,\beta,\mu)$ its left and right unit with respect to the composition is the graph $G_\Delta = {}_\Delta G = (V,\Delta_V,\alpha_\Delta,\beta_\Delta,\mu_\Delta)$ where Δ_V is the set of the identity edges, α_Δ and β_Δ assign to each edge its corresponding vertices, μ_Δ is the constant map, $\mu_\Delta:f \to 1 \in K$ for each $f \in E$.

For a graph $G=(V,E,\alpha,\beta,\mu)$ and a scalar $\delta \in K$ we define the <u>scalar multiplication</u> by the rule

$$\delta \cdot G = (V,E,\alpha,\beta,\delta\cdot\mu) \tag{9}$$

<u>Examples</u>. 1°. The sum let us know whether there exists a connection between two vertices in G_1 or G_2 for $\mathbb{K}=\mathbb{B}$; for $\mathbb{K}=\mathbb{F}$ we have the maximal capacity of this edge; for $\mathbb{K}=\mathbb{P}(Y)$ - the strings in Y^* joining these edges, etc.

2°. For $G_1=G_2$ and $\mathbb{K}=\mathbb{B}$ the composition $G_1 o G_1$ gives all length 2 paths; for $\mathbb{K}=\mathbb{F}$ - their maximal capacity; for $\mathbb{K}=\mathbb{P}(Y)$ - the words of length 2, joining the corresponding vertices; for stochastic graphs - the probability for joining two nodes by a path of length 2.

3°. For a given graph G the graph $G^n=G^{n-1} o G$, $n>1$, is the graph of the n-length paths and the graph $G^{(n)}=\sum_{n=1}^{n} G^p$ is the graph of the paths not longer than n.

Let $G_{AxB}(\mathbb{K})$ be the class of all small graphs with $V=AxB$.

PROPOSITION 3. i) $(G_{AxB}(\mathbb{K}),+,G_0)$ is a commutative monoid;

ii) $(G_{AxB}(\mathbb{K}),+,.,G_0,1)$ is a semimodule over $\mathbb{K}$;

iii) $(G_{AxA}(\mathbb{K}),o,{}_{\Delta}G)$ is a monoid;

iv) $(G_{AxA}(\mathbb{K}),+,o,G_0,{}_{\Delta}G)$ is a semiring.

4. <u>The categories $M(\mathbb{K})$, $R(\mathbb{K})$ and $G(\mathbb{K})$</u>

We define a semimodule category using the ideas [7], [9] for additive and semiadditive categories.

A <u>semimodule structure</u> on a category C consists of functions + and . associating with each pair of parallel arrows $f,g:a \to b$ an arrow $f+g:a \to b$ and for each arrow $h:a \to b$ and $\alpha \in K$ an arrow $\alpha\cdot h:a \to b$ such that the following conditions (SM1), (SM2) and (SM3) are satisfied:

SM1. For each pair of objects (a,b) in C hom(a,b) is a semimodule over the operations + and .;

SM2. The composition law (o) in C is left and right distributive over (+), i.e. whenever

$$c \xrightarrow{k} a \underset{g}{\overset{f}{\rightrightarrows}} b \xrightarrow{h} d$$

are C-morhpisms, it follows that

$$ho(f+g) = hof+hog \quad \text{and} \quad (f+g)ok = fok+gok;$$

SM3. The zero morphisms of C act as monoid identities with respect to (+), i.e. for each C-morphism f,

$$0 + f = f + 0 = f.$$

If (+) and (.) define a semimodule structure on a category C twen we call C is a <u>semimodule category</u> (SM-category).

Let $M(K)$, $R(K)$ and $G(K)$ denote the class of all matrices, relations or graphs respectively over the semiring K.

THEOREM 1. $(M(K),.)$, $(R(K),o)$, $(G(K),o)$ are semimodule categories.

P r o o f. $M(K)$ is a category with morphisms - the matrices $M_{IxJ}:E_{IxI} \to E_{JxJ}$, where E_{IxI} and E_{JxJ} are the square identity matrices, identified with the objects of $M(K)$. The matrix product (2) is a partially defined law of composition in $M(K)$. The functions (+) and (.) are defined by the expressions (1) and (3) respectively. (SM1), (SM2), (SM3) are satisfied according to Prop. 1 and hence $(M(K),.)$ is an SM-category. For $(R(K),o)$ we regard each relation $R \subset AxBxK$ as an arrow; the objects in $R(K)$ are identified with the diagonal relations and the composition of the relations (6) is the composition law in $(R(K),o)$. Each hom-set $R_{AxB}(K)$ is a semimodule over K (Prop. 2 (ii)). For (SM2) let $R_1 \subset AxBxK$, $R_2 \subset AxBxK$ and $R \subset BxDxK$ be given. The sum $R_1 + R_2$ is defined according to (4) and

$$R_1+R_2 = \{(a,b,k'_{ab} + k''_{ab})/(a,b,k'_{ab}) \in R_1 \ \&$$

$$\& \ (a,b,k''_{ab}) \in R_2\} \subset AxBxK.$$

Then the composition $(R_1+R_2)oR$ makes sense and $(R_1+R_2)oR =$ $=\{(a,d,k)/(\forall b\in B)((a,b,k'_{ab})\in R_1)((a,b,k''_{ab})\in R_2)((b,d,k_{bd})\in$ $\in R)(k=\sum_{b\in B}(k'_{ab}+k''_{ab})\cdot k_{bd} = \sum_{b\in B}(k'_{ab}\cdot k_{bd}+k''_{ab}\cdot k_{bd}))\} =$ $=\{(a,d,k)/(\forall b\in B)((a,b,k'_{ab})\in R_1)((b,d,k_{bd})\in R)((a,b,k''_{ab})\in$ $\in R_2)(k=\sum_{b\in B}k'_{ab}\cdot k_{bd}+k''_{ab}\cdot k_{bd})\} = \{(a,d,k)/(\forall b\in B)((a,d,\sum_{b\in B}k'_{ab}\cdot$ $\cdot k_{bd})\in R_1oR)((a,d,\sum_{b\in B}k''_{ab}\cdot k_{bd})\in R_2oR)(k=\sum_{b\in B}k'_{ab}\cdot k_{bd}+k''_{ab}\cdot k_{bd})\} =$ $= \{(a,d,k)/(a,d,k)\in R_1oR+R_2oR\}$. Hence $(R_1+R_2)oR\subset R_1oR+R_2oR$. But $R_1oR+R_2oR=\{(a,d,k'+k'')/(a,d,\sum_{b\in B}k'_{ab}\cdot k_{bd})\in R_1oR)(k'=\sum_{b\in B}k'_{ab}\cdot$ $\cdot k_{bd})((a,d,k''=\sum_{b\in B}k''_{ab}\cdot k_{bd})\in R_2oR)\} = \{(a,d,k'+k'')/$ $/(\forall b\in B)((a,b,k'_{ab})\in R_1)((b,d,k_{bd})\in R)((a,b,k''_{ab}\in R_2)(k'+k''=$ $\sum_{b\in B}(k'_{ab}+k''_{ab})\cdot k_{bd})\} = \{(a,d,k'+k'')/(\forall b\in B)((a,b,k'_{ab}+k''_{ab})\in$ $\in R_1+R_2)((b,d,k_{bd})\in R)(k'+k''=\sum_{b\in B}(k'_{ab}+k''_{ab})\cdot k_{bd})\} =$ $=\{(a,d,k'+k'')/(a,d,k'+k'')\in(R_1+R_2)oR\}$ and thus $(R_1oR+R_2oR)\subset$ $(R_1+R_2)oR$. By analogy we prove $Ro(R_1+R_2)=RoR_1+RoR_2$. (SM3) follows from the existence of the zero morphisms and from Prop. 2 (i). For $(G(K),o)$ the morphisms are the graphs, the law of composition (o) is defined according to (8) and it is partially defined. The rest of the proof follows from Prop. 3.

Having in mind Prop. 1 (i), Prop. 2 (i) and Prop. 3 (i) it is clear that $(M(K),.)$, $(R(K),o)$ and $(G(K),o)$ are not Ab-categories.

COROLLARY 1. If K is a ring, then $(M(K),.)$, $(R(K),o)$ and $(G(K),o)$ are Ab-categories.

We denote by

$R^* = \sum_{k=0}^{\infty} R^k$ - the reflexive and transitive clousure of the relation $R\subset AxAxK$;

$G^* = \sum_{k=0}^{\infty} G^k$ - the free graph over the graph $G=(AxA,E,\alpha,\beta,\mu)$;

$M^* = \sum_{k=0}^{\infty} M^k$ - where M_{AxA} is a square matrix.

Let $R^*(K)$ be the class of all reflexive and transitive closures of the relations over K; $G^*(K)$ be the class of all free graphs over K; $M^*(K)$ be the class of the matrices M^* for all square matrices M_{AxA}.

COROLLARY 2. $R^*(K)$, $G^*(K)$ and $M^*(K)$ are subcategories respectively of the categories $R(K)$, $G(K)$ and $M(K)$.

Considering the connection between the above categories and subcategories, we are needing of some preliminary results.

PROPOSITION 4. The following algebraic structures are isomorphic:

i) The monoids $(M_{AxB}(K),+,0_{AxB})$, $(R_{AxB}(K),+,R_o)$ and $(G_{AxB}(K),+,G_o)$;

ii) The semiring $(G_{AxA}(K),+,.,0_{AxA},E_{AxA})$, $(R_{AxA}(K),+,o,R_o,\Delta_A)$ and $(G_{AxA}(K),+,o,G_o,G_\Delta)$;

iii) The semimodules $(M_{AxB}(K),+,.,0_{AxB},1)$, $(R_{AxB}(K),+,.,R_o,1)$ and $(G_{AxB}(K),+,.,G_o,1)$.

P r o o f. i) Let $h_{AxB}=h:R_{AxB}(K) \to M_{AxB}(K)$ be the following map: $h(R)=M_{AxB}=(m_{ab})$, where $m_{ab}=pr_3(a,b,k_{ab})=k_{ab}$. The direct verification shows that h is a bijection. Since $h(R_o)=0_{AxB}$ and $h(R_1+R_2) = h\{(a,b,k'_{ab}+k''_{ab})\} = (k'_{ab}+k''_{ab})_{AxB} = (k'_{ab})_{AxB}+(k''_{ab})_{AxB} = h(R_1)+h(R_2)$ we obtain $(M_{AxB}(K),+,0_{AxB}) \approx (R_{AxB}(K),+,R_o)$. We may prove the isomorphism $(G_{AxB}(K),+,G_o) \approx (M_{AxB}(K),+,0_{AxB})$ using the map $g_{AxB}=g:G=(AxB,E,\alpha,\beta,\mu) \to M_{AxB}=(m_{ab})$, where $m_{ab}=pr_3(\alpha(f),\beta(f),\mu(f))$ for $f \in E$ and $\alpha(f)=a$, $\beta(f)=b$.

ii) Using the same notations as in (i) for A=B we have: $h(\Delta_A)=E_{AxA}$, $h(R_1 o R_2)=M_{R_1 o R_2}=M_{R_1}\cdot M_{R_2}=h(R_1)\cdot h(R_2)$ and $h((R_1+R_2)oR) = h(R_1)\cdot h(R)+h(R_2)\cdot h(R)$, i.e. we obtain the first isomorphism.

By analogy we can prove the rest of the statement.

If C and D are SM-categories, a functor $T:C \to D$ is said to be an <u>SM-functor</u> when every function $T:C(a,a') \to D(T(a),T(a'))$ is a homomorphism of semimodules. If T is an isomorphism then it is called <u>SM-isomorphism</u>.

THEOREM 2. The categories $(M(\mathbb{K}),.)$, $(R(\mathbb{K}),o)$ and $(G(\mathbb{K}),o)$ are SM-isomorphic.

P r o o f. Let $H:R(\mathbb{K}) \to M(\mathbb{K})$ be a functor defined by the map $h=h_{AxB}:R_{AxB}(\mathbb{K}) \to M_{AxB}(\mathbb{K})$ (see Prop. 4) for arbitrary sets A,B as follows: for each relation $R \subset AxBxK$ we have $H(R)=h(R)=M_{AxB}$ and for each object R_0 is valid $H(R_0)=h(R_0)=E$. According to Prop. 4 (i) H is an isomorphism and $H(R'oR'')=h(R'oR'')=H(R')\cdot H(R'')$. Since $R(\mathbb{K})$ and $M(\mathbb{K})$ are SM-categories (Th. 1) and H is an isomorphism as a functor from $R(\mathbb{K})$ to $M(\mathbb{K})$ then these categories are SM-isomorphism. Extending the map g from Prop. 4 (ii) we can construct a functor $G:G(\mathbb{K}) \to M(\mathbb{K})$. It follows from Prop. 4 (ii) and Th. 1 that G is an SM-isomorphism.

COROLLARY 3. The categories $M^*(\mathbb{K})$, $R^*(\mathbb{K})$ and $G^*(\mathbb{K})$ are isomorphic.

Obviously instead of computing with graphs or relations we can use matrices and corresponding operations.

COROLLARY 4. For each bounded chain $\mathbb{K}=\mathbb{L}$ and a finite set A with cardinality $|A|=n$ holds:

i) $G^* = \sum_{k=0}^{n-1} G^k$ for each graph $G=(V,E,\alpha,\beta,\mu)$ with $V \subset AxA$;

ii) $R^* = \sum_{k=0}^{n-1} R^k$ for each relation $R \subset AxAxK$;

iii) $M^* = \sum_{k=0}^{n-1} M^k$ for each matric M_{AxA}.

P r o o f. i) for $\mathbb{K}=\mathbb{F}$ see [8]. The generalization for $\mathbb{K}=\mathbb{L}$ is not difficult; (ii) and (iii) follow from (i), Th. 2 and Cor. 3.

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