SOME RELATIONS BETWEEN JACOBI AND BESSEL POLYNOMIALS

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Abstract

Using a classical method, the expansion of an orthogonal polynomial in a series of other system of orthogonal polynomials, is given.

1. The purpose of this paper is to give the expression of the Jacobi polynomial through Bessel polynomials and vice versa.

The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(-1)^n(1+\beta)_n}{n!}\,{}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\beta; \frac{1+x}{2}\right) \quad (1)$$

where

$${}_2F_1\,(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n x^n}{(c)_n\, n!}$$

is the hypergeometric function, and

$$(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\dots(\alpha+n-1)\,, \quad n \le 1$$

$$(\alpha)_0 = 1\,, \quad \alpha \ne 0$$

the factorial function [1].

Similarly the generalized hypergeometric function is ${}_pF_q\,(\alpha_1, \alpha_2, \dots, \alpha_p; \beta_1, \beta_2, \dots, \beta_q; x)$.

Replacing x by $2x-1$, the equation (1) yields the form

$$\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(2x-1)}{(1+\beta)_n} = \sum_{s=0}^{n} \frac{(-1)^{n-s}(1+\alpha+\beta)_{n+s}x^s}{s!\,(n-s)!\,(1+\beta)_s} \quad (2)$$

and the property

$$x^n = (1+\alpha)_n \sum_{k=0}^{n} \frac{(-1)^k(-n)_k(1+\alpha+\beta+2k)(1+\alpha+\beta)_k P_k^{(\beta,\alpha)}(2x-1)}{(1+\alpha+\beta)_{n+k+1}(1+\alpha)_k}\,.$$

The polynomial

$$\varphi_n(c,x) = \frac{(c)_n}{n!}\,{}_2F_0\,(-n,\, c+n;\, -;\, x) = \sum_{k=0}^{n} \frac{(-1)^k(c)_{n+k}x^k}{k!\,(n-k)!} \quad (3)$$

included the simple Bessel polynomial

$$y_n(x) = {}_2F_0\left(-n, 1+n; -; -\frac{x}{2}\right) = \varphi_n\left(1, -\frac{x}{2}\right)$$

and the generalized one

$$y_n(a,b,x) = {}_2F_0\left(-n,\, a-1+n;\, - :\, -\frac{x}{b}\right) = \frac{n!}{(a-1)_n}\,\varphi_k\left(a-1, -\frac{x}{b}\right)$$

as particular cases.

The equation (3) yields the property

$$x^n = n!\sum_{k=0}^{n} \frac{(-1)^k (c+2k)\varphi_k(c,x)}{(n-k)!\,(c)_{n+k+1}}. \tag{4}$$

2. Let us consider the series

$$\begin{aligned}
&\sum_{n=0}^{\infty} \varphi_n(c,x)t^n \\
&= \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s (c)_{n+s} x^s t^n}{s!\,(n-s)!} \\
&= \sum_{n,s=0}^{\infty} \frac{(-1)^s (c)_{n+2s} x^s t^{n+s}}{s!\,n!} \\
&= \sum_{n,s=0}^{\infty}\sum_{k=0}^{s} \frac{(-1)^s (1+\alpha)_s (c)_{n+2s} (1+\alpha+\beta+2k)(1+\alpha+\beta)_k P_k^{(\beta,\alpha)}(2x-1)t^{n+s}}{n!\,(s-k)!(1+\alpha+\beta)_{s+k+1}(1+\alpha)_k} \\
&= \sum_{n,k,s=0}^{\infty} \frac{(-1)^{s+k} (1+\alpha)_{s+k} (c)_{n+2s+2k} (1+\alpha+\beta+2k)(1+\alpha+\beta)_k P_k^{(\beta,\alpha)}(2x-1)t^{n+s+k}}{s!\,n!\,(1+\alpha+\beta)_{s+2k+1}(1+\alpha)_k}
\end{aligned}$$

in which we have used the identity

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n+k) \tag{5}$$

to collect powers of t in the last summation above.

By the same identity used conversely, we may write

$$\begin{aligned}
&\sum_{k=0}^{\infty} \varphi_n(c,x)t^n \\
&= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^{s+k} (1+\alpha)_{s+k} (c)_{n+2k+s} (1+\alpha+\beta+2k)(1+\alpha+\beta)_k P_k^{(\beta,\alpha)}(2x-1)t^{n+k}}{s!(n-s)!(1+\alpha+\beta)_{s+2k+1}(1+\alpha)_k} \\
&= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^k (-n)_s (1+\alpha+k)_s (c+n+2k)_s (c)_{n+2k} P_k^{(\beta,\alpha)}(2x-1)t^{n+k}}{n!\,s!\,(2+\alpha+\beta+2k)_s(1+\alpha+\beta+k)_k} \\
&= \sum_{n,k=0}^{\infty} {}_3F_1\left(\begin{matrix}-n, c+n+2k, 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\; 1\right) \frac{(-1)^k (c)_{n+2k} P_k^{(\beta,\alpha)}(2x-1)t^{n+k}}{n!\,(1+\alpha+\beta+k)_k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k, c+n+k, 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\; 1\right) \frac{(-1)^k (c)_{n+k} P_k^{(\beta,\alpha)}(2x-1)t^{n}}{(n-k)!\,(1+\alpha+\beta+k)_k}.
\end{aligned}$$

Therefore

$$\varphi_n(c,x) = \sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k, c+n+k, 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\; 1\right) \frac{(-1)^k (c)_{n+k} P_k^{(\beta,\alpha)}(2x-1)}{(n-k)!(1+\alpha+\beta+k)_k}.$$

For the simple Bessel polynomial we have

$$y_n(x)=\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k,\ 1+n+k,\ 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\ 1\right)\frac{(n+k)!\,P_k^{(\alpha,\beta)}(x+1)}{(n-k)!\,(1+\alpha+\beta+k)_k}$$

or

$$y_n(x)=\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k, 1+n+k, 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\ -\frac{1}{2}\right)\frac{(n+k)!\,P_k^{(\beta,\alpha)}(2x-1)}{2^k(n-k)!\,(1+\alpha+\beta+k)_k}$$

and for the generalized one

$$y_n(a,b.x)$$

$$=\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k, a-1+n+k, 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\ 1\right)\frac{n!\,(a+n-1)_k P_k^{(\alpha,\beta)}\left(\frac{2x}{b}+1\right)}{(n-k)!\,(1+\alpha+\beta+k)_k}.$$

or

$$y_n(a,b,x)$$

$$=\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k, a-1+n+k, 1+\alpha+k; \\ 2+\alpha+\beta+2k;\end{matrix}\ -\frac{1}{b}\right)\frac{n!(a+n-1)_k P_k^{(\beta,\alpha)}(2x-1)}{b^k(n-k)!(1+\alpha+\beta+k)_k}.$$

3. Next let us expand the Jacobi polynomial in a series of the polynomials $\varphi_n(c,x)$, using the relations (2) and (4).

We start from the series

$$\theta(x,t)=\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(2x-1)}{(1+\beta)_n}.$$

Then by (2) we have

$$\begin{aligned}\theta(x,t)&=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^{n-s}(1+\alpha+\beta)_{n+s}x^s t^n}{s!\,(n-s)!\,(1+\beta)_s}\\ &=\sum_{n,s=0}^{\infty}\frac{(-1)^n(1+\alpha+\beta)_{n+2s}x^s t^{n+s}}{s!\,n!\,(1+\beta)_s}\\ &=\sum_{n,s=0}^{\infty}\sum_{k=0}^{s}\frac{(-1)^{n+k}(1+\alpha+\beta)_{n+2s}(c+2k)\varphi_k(c,x)t^{n+s}}{n!\,(s-k)!\,(c)_{s+k+1}(1+\beta)_s}\\ &=\sum_{n,s,k=0}^{\infty}\frac{(-1)^{n+k}(1+\alpha+\beta)_{n+2s+2k}(c+2k)\varphi_k(c,x)t^{n+s+k}}{n!\,s!\,(c)_{s+2k+1}(1+\beta)_{s+k}}\end{aligned}$$

in which we have used the indentity (5) again.

Applying the same identity conversely, we obtain

$$\theta(x,t)$$

$$=\sum_{n,k=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^{n+k+s}(1+\alpha+\beta+n+2k)_s(-n)_s(1+\alpha+\beta)_{n+2k}\varphi_k(c,x)t^{n+k}}{n!\,s!\,(1+\beta+k)_s(c+2k+1)_s(c)_{2k}(1+\beta)_k}$$

$$=\sum_{n,k=0}^{\infty} {}_2F_2\left(\begin{matrix}-n, 1+\alpha+\beta+n+2k; \\ 1+\beta+k, c+2k+1;\end{matrix}\ 1\right)\frac{(-1)^{n+k}(1+\alpha+\beta)_{n+2k}\varphi_k(c,x)t^{n+k}}{n!\,(1+\beta)_k(c)_{2k}}$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_2F_2\left(\begin{matrix}-n+k,1+\alpha+\beta+n+k;\\ c+2k+1,1+\beta+k;\end{matrix}\;1\right)\frac{(-1)^n(1+\alpha+\beta)_{n+k}\varphi_k(c,x)t^{n+k}}{(n-k)!\,(1+\beta)_k(c)_{2k}}.$$

Therefore

$$P_n^{(\alpha,\beta)}(2x-1)$$
$$=\frac{(1+\beta)_n}{(1+\alpha+\beta)_n}\sum_{a=0}^{n} {}_2F_2\left(\begin{matrix}-n+k,1+\alpha+\beta+n+k;\\ c+2k+1,1+\beta+k;\end{matrix}\;1\right)\frac{(-1)^n(1+\alpha+\beta)_{n+k}\varphi_k(c,x)}{(n-k)!\,(c)_{2k}(1+\beta)_k}.$$

For the simple Bessel polynomial we have the relation

$$P_n^{(\alpha,\beta)}(x+1)$$
$$=\sum_{k=0}^{n} {}_2F_2\left(\begin{matrix}-n+k,1+k+n+\alpha+\beta;\\ 1+\alpha+k,2+2k;\end{matrix}\;1\right)\frac{(1+\alpha)_n(1+n+\alpha+\beta)_k y_k(x)}{(n-k)!\,(2k)!\,(1+\alpha)_k}$$

and for generalized one

$$P_n^{(\alpha,\beta)}\left(\frac{2x}{b}+1\right)$$
$$=\sum_{k=0}^{n} {}_2F_2\left(\begin{matrix}-n+k,1+k+n+\alpha+\beta;\\ 1+\alpha+k,a+2k;\end{matrix}\;1\right)\frac{(1+\alpha)_n(1+n+\alpha+\beta)_k y_k(a,b,x)}{k!\,(n-k)!\,(a+k-1)_k\,(1+\alpha)_k}.$$

4. In the special case $\alpha=\beta=0$ for the relations between the Legendre and Bessel polynomials we obtain

$$y_n(x)=\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k,1+k+n,1+k;\\ 2+2k;\end{matrix}\;1\right)\frac{(n+k)!\,k!\,P_k(x+1)}{(n-k)!\,(2k)!},$$

$$y_n(a,b,x)$$
$$=\sum_{k=0}^{n} {}_3F_1\left(\begin{matrix}-n+k,a-1+k+n,1+k;\\ 2+2k;\end{matrix}\;1\right)\frac{(a+n-1)_k n!\,k!P_k\left(\frac{2x}{b}+1\right)}{(n-k)!\,(2k)!}$$

and

$$P_n(x+1)=\sum_{k=0}^{n} {}_2F_2\left(\begin{matrix}-n+k,n+k+1;\\ 1+k,2+2k;\end{matrix}\;1\right)\frac{(n+k)!\,y_k(x)}{k!\,(n-k)!\,(2k)!},$$

$$P_n\left(\frac{2x}{b}+1\right)=\sum_{k=0}^{n} {}_2F_2\left(\begin{matrix}-n+k,1+k+n;\\ 1+k,a+2k;\end{matrix}\;1\right)\frac{(n+k)!y_k(a,b,x)}{k!(n-k)!(a+k-1)_k k!}.$$

References

[1] Rainville E. D.: *Special Functions*, The Macmillan Company, New York, 1960.

[2] Szegö G.: *Orthogonal Polynomials*, American Mathematical Society, New York, 1950.

НЕКОИ РЕЛАЦИИ МЕЃУ ПОЛИНОМИТЕ НА JACOBI И BESSEL

Резиме

Во трудот е дадено развивање на ортогонален полином од еден ортогонален систем во ред од полиноми од друг ортогонален систем.