RELATIONS BETWEEN JACOBI POLYNOMIALS OF DIFFERENT PARAMETERS

Прилози МАНУ, Оддел. за мат.-тех. науки, XX/1-2, 1999, 45-50

A b s t r a c t: Some relations of Jacobi and related polynomials are given.

1. The Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ defined by

$$P_n^{(\alpha,\beta)}(x)=\frac{(1+\alpha)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,1+\alpha+\beta+n;\ \frac{1-x}{2}\\ 1+\alpha;\end{matrix}\right),\ \alpha>-1,\ \beta>-1$$

yields a finite form

$$P_n^{(\alpha,\beta)}(x)=\sum_{s=0}^{n}\frac{(-1)^s(1+\alpha)_n(1+\alpha+\beta)_{n+s}}{s!(n-s)!(1+\alpha)_s(1+\alpha+\beta)_n}\left(\frac{1-x}{2}\right)^s. \tag{1}$$

It has the property

$$\left(\frac{1-x}{2}\right)^n=(1+\gamma)_n\sum_{k=0}^{n}\frac{(-1)^k n!(1+\gamma+\delta+2k)(1+\gamma+\delta)_k P_k^{(\gamma,\delta)}(x)}{(n-k)!(1+\gamma+\delta)_{n+1+k}(1+\gamma)_k}. \tag{2}$$

2. Consider the series

$$\Psi(x,t)=\sum_{n=0}^{\infty}\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)t^n}{(1+\alpha)_n}. \tag{3}$$

Using (1) we obtain

$$\Psi(x,t)=\sum_{n=0}^{\infty}\sum_{s=0}^{n}\frac{(-1)^s(1+\alpha+\beta)_{n+s}\left(\frac{1-x}{2}\right)^s t^n}{s!(n-s)!(1+\alpha)_s}$$

$$=\sum_{n,s=0}^{\infty}\frac{(-1)^s(1+\alpha+\beta)_{n+2s}\left(\frac{1-x}{2}\right)^s t^{n+s}}{s!\,n!(1+\alpha)_s}.$$

By (2) we may write

$$\Psi(x,t)$$

$$=\sum_{n,s=0}^{\infty}\sum_{k=0}^{s}\frac{(-1)^{s+k}(1+\alpha+\beta)_{n+2s}(1+\gamma+\delta+2k)(1+\gamma+\delta)_k(1+\gamma)_s P_k^{(\gamma,\delta)}(x)t^{n+s}}{n!(1+\alpha)_s(s-k)!(1+\gamma+\delta)_{s+1+k}(1+\gamma)_k}$$

$$=\sum_{n,s,k=0}^{\infty}\frac{(-1)^s(1+\alpha+\beta)_{n+2s+2k}(1+\gamma+\delta+2k)(1+\gamma+\delta)_k(1+\gamma)_{s+k}P_k^{(\gamma,\delta)}(x)t^{n+s+k}}{n!s!(1+\alpha)_{s+k}(1+\gamma+\delta)_{s+2k+1}(1+\gamma)_k}$$

in which we have used the identity [1], [2]

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}A(k,n)=\sum_{n,k=0}^{\infty}A(k,n+k)$$

to collect the powers of t.

Key words: Jacobi polynomials, expansion

By the same identity used inversely we write

$\Psi(x,t)$

$$= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s(1+\alpha+\beta)_{n+s+2k}(1+\gamma+\delta+2k)(1+\gamma+k)_s(1+\gamma+\delta)_k P_k^{(\gamma,\delta)}(x)t^{n+k}}{(n-s)!s!(1+\alpha+k)_s(2+\gamma+\delta+2k)_s(1+\alpha)_k(1+\gamma+\delta)_{2k+1}}$$

$$= \sum_{n,k=0}^{\infty} {}_3F_2\left(\begin{matrix} -n, 1+\alpha+\beta+n+2k, 1+\gamma+k; \\ 1+\alpha+k, 2+\gamma+\delta+2k; \end{matrix} 1\right)\frac{(1+\alpha+\beta)_{n+2k}P_k^{(\gamma,\delta)}(x)t^{n+k}}{n!(1+\alpha)_k(1+\gamma+\delta+k)_k}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_3F_2\left(\begin{matrix} -n+k, 1+\alpha+\beta+n+k, 1+\gamma+k; \\ 1+\alpha+k, 2+\gamma+\delta+2k; \end{matrix} 1\right)\frac{(1+\alpha+\beta)_{n+k}P_k^{(\gamma,\delta)}(x)t^{n}}{(n-k)!(1+\alpha)_k(1+\gamma+\delta+k)_k}$$

Then by (3) we conclude that

$$\frac{(1+\alpha+\beta)_n P_n^{(\alpha,\beta)}(x)}{(1+\alpha)_n}$$

$$= \sum_{k=0}^{n} {}_3F_2\left(\begin{matrix} -n+k, 1+\alpha+\beta+n+k, 1+\gamma+k; \\ 1+\alpha+k, 2+\gamma+\delta+2k; \end{matrix} 1\right)\frac{(1+\alpha+\beta)_{n+k}P_k^{(\gamma,\delta)}(x)}{(n-k)!(1+\alpha)_k(1+\gamma+\delta+k)_k}$$

or

$P_n^{(\alpha,\beta)}(x)$

$$= \sum_{k=0}^{n} {}_3F_2\left(\begin{matrix} -n+k, 1+\alpha+\beta+n+k, 1+\gamma+k; \\ 1+\alpha+k, 2+\gamma+\delta+2k \end{matrix} 1\right)\times \frac{(1+\alpha+\beta+n)_k(1+\alpha+k)_{n-k}P_k^{(\gamma,\delta)}(x)}{(n-k)!(1+\gamma+\delta+k)_k} \tag{4}$$

which is an expansion of Jacobi polynomial $P_k^{(\alpha,\beta)}(x)$ is a series of the polynomials $P_n^{(\gamma,\delta)}(x), n=0,1,2\dots$.

3. *Special cases.* We shall consider now some relations in special cases using the relation (4) and obtained for $\beta=\alpha$, so called the ultraspherical polynomial $P_n^{(\alpha,\alpha)}(x)$ and the cases $\alpha=\beta=0$ i.e. $P_n^{(0,0)}(x)=P_n(x)$ – Legendre polynomial.

It is known [1] that

$${}_3F_2\left(\begin{matrix} -n, a, b; \\ c, 1-c+a+b-n; \end{matrix} 1\right)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}.$$

Applying this to (4) we obtain

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(\alpha-\gamma)_{n-k}(1+\beta+k)_{n-k}(1+\alpha+\beta+n)_k P_k^{(\gamma,\beta)}(x)}{(n-k)!(2+\gamma+\beta+2k)_{n-k}(1+\gamma+\beta+k)_k}$$

or

$P_n^{(\alpha,\beta)}(x)$

$$= \sum_{k=0}^{n}\frac{(\alpha-\gamma)_{n-k}(1+\beta+k)_{n-k}(1+\alpha+\beta+n)_k(1+\gamma+\beta+2k)P_k^{(\gamma,\beta)}(x)}{(n-k)!(1+\gamma+\beta+k)_{n+1}} \tag{5}$$

It we put $\gamma=\beta$, from (5) we have

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(\alpha-\beta)_{n-k}(1+\beta+k)_{n-k}(1+\alpha+\beta+n)_k(1+2\beta+2k)P_k^{(\beta,\beta)}(x)}{(n-k)!(1+2\beta+k)_{n+1}}. \tag{6}$$

Similarly, it we put in (4) $\alpha=\beta$ we find

$$P_n^{(\alpha,\alpha)}(x)$$
$$=\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,1+2\alpha+n+k,1+\gamma+k;\\ 1+\alpha+k,2+\gamma+\delta+2k;\end{matrix}1\right)\times$$
$$\frac{(1+2\alpha+n)_k(1+\alpha+k)_{n-k}P_k^{(\gamma,\delta)}(x)}{(n-k)!(1+\gamma+\delta+k)_k} \tag{7}$$

or by (5)

$$P_n^{(\alpha,\alpha)}(x)$$
$$=\sum_{k=0}^{n}\frac{(\alpha-\gamma)_{n-k}(1+\alpha+k)_{n-k}(1+2\alpha+n)_k(1+\alpha+\gamma+2k)P_k^{(\gamma,\alpha)}(x)}{(n-k)!(1+\alpha+\gamma+k)_{n+1}}. \tag{8}$$

As particular cases from (5) and (8) we find

$$P_n^{(\alpha,\beta)}(x)=\sum_{k=0}^{n}\frac{(\alpha)_{n-k}(1+\alpha+\beta+n)_k(1+\beta+2k)P_k^{(0,\beta)}(x)}{(n-k)!(1+\beta+n)_{k+1}} \tag{5a}$$

and

$$P_n^{(\alpha,\alpha)}(x)=\sum_{k=0}^{n}\frac{(\alpha)_{n-k}(1+2\alpha+n)_k(1+\alpha+2k)P_k^{(0,\alpha)}(x)}{(n-k)!(1+\alpha+n)_{k+1}}. \tag{8a}$$

If $\gamma=\delta=0$, from (4) we have

$$P_n^{(\alpha,\beta)}(x)$$
$$=\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,1+\alpha+\beta+n+k,1+k;\\ 1+\alpha+k,2+2k\end{matrix}1\right)\frac{(1+\alpha+\beta+n)_k(1+\alpha+k)_{n-k}P_k(x)}{(n-k)!(1+k)_k} \tag{9}$$

and if $\alpha=\beta=0$, it is

$$P_n(x)=\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,1+n+k,\ 1+\gamma+k;\\ 1+k,2+\gamma+\delta+2k;\end{matrix}1\right)\frac{(1+k)_nP_k^{(\gamma,\delta)}(x)}{(n-k)!(1+\gamma+\delta+k)_k} \tag{10}$$

Similarly, from (7) we find

$$P_n^{(\alpha,\alpha)}(x)=$$
$$=\sum_{k=0}^{n}{}_3F_2\left(\begin{matrix}-n+k,1+2\alpha+n+k,\ 1+k;\\ 1+\alpha+k,2+2k;\end{matrix}1\right)\frac{(1+2\alpha+n)_k(1+\alpha+k)_{n-k}P_k(x)}{(n-k)!(1+k)_k} \tag{11}$$

and

$$P_n(x) = \sum_{k=0}^{n} {}_3F_2\left(\begin{matrix} -n+k, 1+n+k, \ 1+\gamma+k; \\ 1+k, 2+2\gamma+2k; \end{matrix} 1\right) \frac{(1+k)_n P_k^{(\gamma,\gamma)}(x)}{(n-k)!(1+2\gamma+k)_k} \tag{12}$$

4. *Application.* Using

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{(\alpha,\beta)}(x) P_n^{(\alpha,\beta)}(x)dx$$

$$= \frac{2^{1+\alpha+\beta}\Gamma(1+\alpha+n)\,\Gamma(1+\beta+n)}{n!(1+\alpha+\beta+2n)\,\Gamma(1+\alpha+\beta+n)}. \qquad m=n$$

$$=0, \qquad m \neq n$$

by (5) we obtain

$$\int_{-1}^{1} (1-x)^{\gamma}(1+x)^{\delta} P_n^{(\alpha,\beta)}(x) P_m^{(\gamma,\delta)}(x)dx$$

$$= {}_3F_2\left(\begin{matrix} m-n, 1+\alpha+\beta+n+m, 1+\gamma+m; \\ 1+\alpha+m, 2+\gamma+\gamma+2m; \end{matrix} 1\right)\times$$

$$\frac{2^{1+\gamma+\delta}}{m!(n-m)!}(1+\alpha+\beta+n)_m (1+\alpha+m)_{n-m} B(1+\gamma+m, 1+\delta+m)$$

if $0 < m \leq n$ and the integral in zero for $m > n$. In special case [1]

$$\int_{-1}^{1} (1-x^2)^{\beta} P_n^{(\alpha,\beta)}(x) P_m^{(\beta,\beta)}(x)dx$$

$$= \frac{2^{1+2\beta}(\alpha-\beta)_{n-m}(1+\alpha+\beta+n)_m B(1+\beta+n, 1+\beta+m)}{n!(n-m)!}$$

where $B(p,q)$ is Beta function defined by

$$B(p,q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}, \quad \mathrm{Re}(p) > 0, \quad \mathrm{Re}(q) > 0\,.$$

REFERENCES

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Резиме

РЕЛАЦИИ МЕЃУ JACOBI-ЕВИ ПОЛИНОМИ СО РАЗЛИЧНИ ПАРАМЕТРИ

Во трудот се даваат некои релации на полиномите на Jacobi.