SOME RELATIONS OF BESSEL POLYNOMIALS

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Abstract

The relations of Bessel polynomials of different arguments and parameters are given.

1. The simple Bessel polynomial [1] is defined by

$$y_n(x) = {}_2F_0\left(-n, 1+n; -; -\frac{1}{2}x\right)$$

and the generalised one by

$$y_n(a,b,x) = {}_2F_0\left(-n, a-1+n; -; -\frac{x}{b}\right).$$

Let us take as a general form [1]

$$\varphi_n(c,x) = \frac{(c)_n}{n!}\, {}_2F_0\,(-n, c+n; -; x)$$

for polynomials of this character. They may be write as

$$\varphi_n(c,x) = \sum_{s=0}^{n} \frac{(-1)^s (c)_{n+s} x^s}{s!\,(n-s)!}$$

and have the property

$$x^s = s! \sum_{k=0}^{s} \frac{(-1)^k (c+2k)\varphi_k(c,x)}{(s-k)!\,(c)_{s+k+1}}.$$

2. Consider the series

$$\sum_{n=0}^{\infty} \varphi_n(c,xy)t^n = \sum_{n=0}^{\infty}\sum_{s=0}^{n} \frac{(-1)^s (c)_{n+s} x^s y^s t^n}{s!\,(n-s)!}$$

$$= \sum_{n,s=0}^{\infty} \frac{(-1)^s (c)_{n+2s} x^s y^s t^{n+s}}{s!\,n!}$$

$$= \sum_{n,s=0}^{\infty}\sum_{k=0}^{s} \frac{(-1)^s (c)_{n+2s} y^s (a+2k)\varphi_k(a,x) t^{n+s}}{n!\,(s-k)!\,(a)_{s+k+1}}$$

$$= \sum_{n,k,s=0}^{\infty} \frac{(-1)^s (c)_{n+2s+2k} y^{s+k} (a+2k)\varphi_k(a,x) t^{n+s+k}}{n!\,s!\,(a)_{s+2k+1}}.$$

Now rearrange terms in the above. We obtain

$$\sum_{n=0}^{\infty} \varphi_n(c,xy)t^n$$

$$= \sum_{n,k=0}^{\infty}\sum_{s=0}^{n} \frac{(-n)_s (c+n+2k)_s y^{s+k} (c)_{n+2k} (a+2k)\varphi_k(a,x) t^{n+k}}{n!\,s!\,(a+2k+1)_s (a)_{2k+1}}$$

$$= \sum_{n,k=0}^{\infty} {}_2F_1\left(\begin{matrix}-n,\ c+n+2k;\\ a+2k+1;\end{matrix}\ y\right) \frac{y^k (c)_{n+2k}\varphi_k(a,x) t^{n+k}}{n!\,(a)_{2k}}$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} {}_2F_1\left(\begin{matrix}-n+k,\ c+n+k;\\ a+2k+1;\end{matrix}\ y\right) \frac{y^k (c)_{n+k}\varphi_k(a,x) t^{n}}{(n-k)!\,(a)_{2k}}.$$

Finally

$$\varphi_n(c,xy) = \sum_{k=0}^{n} {}_2F_1\left(\begin{matrix}-n+k,\ c+n+k;\\ a+2k+1;\end{matrix}\ y\right) \frac{y^k (c)_{n+k}\varphi_k(a,x)}{(n-k)!\,(a)_{2k}}. \qquad (1)$$

3. *Special cases*

1°. If we put $y = 1$ in (1), we obtain

$$\varphi_n(c,x) = \sum_{k=0}^{n} {}_2F_1\left(\begin{matrix} -n+k,\, c+n+k; \\ a+2k+1; \end{matrix}\; 1\right) \frac{(c)_{n+k}\varphi_k(a,x)}{(n-k)!\,(a)_{2k}}\,.$$

Because

$${}_2F_1\left(\begin{matrix} -n+k,\, c+n+k; \\ a+2k+1; \end{matrix}\; 1\right) = \frac{(a-c-n+k+1)_{n-k}}{(a+2k+1)_{n-k}}\,,$$

we have

$$\varphi_n(c,x) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(c-a)_{n-k}(c)_{n+k}(a+2k)\varphi_k(a,x)}{(n-k)!\,(a)_{n+k+1}}\,. \tag{2}$$

2°. From (2) we have: for $a = 1$

$$\varphi_n\left(c, -\frac{x}{2}\right) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(c-1)_{n-k}(c)_{n+k}(2k+1)y_k(x)}{(n-k)!\,(n+k+1)!}$$

and for $c = 1$

$$y_n(x) = \sum_{k=0}^{n} \frac{(a-1)_{n-k}(n+k)!\,(a+2k)\varphi_k\left(a, -\frac{x}{2}\right)}{(n-k)!\,(a)_{n+k+1}}\,.$$

3°. Similarly, the realtion between the generalised Bessel polynomials of different parameters is given by

$$y_n(c,b,x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n}{k} \frac{(c-a-1)_{n-k}(c+n-1)_k(a+2k-1)y_k(a,b,x)}{(a+k-1)_{n+1}}\,. \tag{3}$$

If we replace x by $-\frac{x}{2}$ and y by $\frac{2}{b}$, from (1) we obtain

$$y_n(c,b,x) = \sum_{k=0}^{n} {}_2F_1\left(\begin{matrix} -n+k,\, c+n+k-1; \\ 2+2k; \end{matrix}\; \frac{2}{b}\right)\binom{n}{k} \frac{(c+n-1)_k y_k(x)}{b^k(2k-1)!!}\,.$$

4. *Application.* It is known that

$$\frac{1}{2\pi i}\int_C e^{-2/x} y_m(x)y_n(x)dx = (-1)^{n+1}\frac{2}{2n+1}\delta_{mn}\,. \quad i = \sqrt{-1}$$

where $C\colon |x| = 1$ and δ_{mn} is Kronecker's delta.

Then

$$\frac{1}{2\pi i}\int_C e^{-2/x} y_n(a,b,x)y_m(x)dx = (-1)^{m+1}\binom{n}{m}\frac{(a+n-1)_m}{b^m(2m+1)!!}\,{}_2F_1\left(\begin{matrix} m-n,\, a+n+m-1; \\ 2m+2; \end{matrix}\; \frac{2}{b}\right)\,. \tag{4}$$

References

[1] Rainville E. D.: *Special Functions*, The Macmillan Company, New York (1960).

[2] Krall H. L. and Frink O. A.: *A New Class of Orthogonal Polynomials: the Bessel Polynomials*, Trans. American Mathematical Society, **65**, p. 110-115 (1942).

[3] Popov B. S.: *Some relations between Bessel and Bernoulli polynomials*, Annuaire de l'institut des mathematiques, Skopje, Macedoine, T. **38**, (1997).

НЕКОИ РЕЛАЦИИ НА ПОЛИНОМИТЕ НА БЕСЕЛ

Резиме

Се даваат релации за полиноми на Бесел со различни аргументи и параметри.