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MULTIPLE EULER – MACLAURIN SUMMATION FORMULA

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Abstract

d-dimensional finite Euler-Maclaurin summation formula is derived for sequences which are restrictions of coordinatewise once differentiable nonnegative real functions.

1. In this short note we will extend it to the closed integral expression for

$$\sum_{\ell=1}^{d}\sum_{j_\ell=0}^{n_\ell} a_{\mathbf{j}}, \tag{1}$$

where d-dimensional sequence $\{a_{\mathbf{j}}\,:\ \mathbf{j}=(j_1,\cdots,j_d),\ 0\le j_l\le n_l\}$ is a restriction of function $a_{\mathbf{x}}=a(x_1,\cdots,x_d),\ d\ge 2$ to $\mathbf{N}_0^d$, such that

$$\frac{\partial^d a}{\partial x_1\cdots\partial x_d}\in C\big([0,n_1]\times\cdots\times[0,n_d]\big), \tag{2}$$

i.e. $a_{\mathbf{x}}$ is continuously differentiable once with respect $\mathbf{x}=(x_1,\cdots,x_d)$.

In this respect consider the well-known Euler-Maclaurin summation formula

$$\sum_{j=0}^{n} a_j=\int_0^n a(u)du+\frac{1}{2}(a_n+a_0)+\int_0^n a'(u)B_1(u)du, \tag{3}$$

valid for $a_x=a(x)\in C^1[0,\infty)$, [3, **296**, p.539]. Here $B_1(u)=\{u\}-\frac{1}{2}$ is the Bernoulli polynomial of first degree, while $\{u\}$ denotes the fractional part of u in the whole article. At this point one mentions the operator form variant of previous formula (which proof is indeed straigthforward), such that will be used in the sequel. Namely, it holds true

$$\sum_{j=0}^{n} a_j = a_0 + \int_0^n (a(u) + \{u\}a'(u))du \equiv a_0 + \int_0^n \partial_u^* a(u)du, \tag{4}$$

where

$$\partial_u^* := 1 + \{u\}\frac{\partial}{\partial u}. \tag{5}$$

2. Now, we give the d-fold Euler-Maclaurin summation formula for (1). At first we introduce some necessary conventions. Let us denote

$$a(\mathbf{x})\Big|_{x_m=0,\ m\in\{1,\cdots,d\}\backslash\{j_1,\cdots,j_k\}} := a(x_{j_1},\cdots,x_{j_k}), \qquad a_0 \equiv a(0,\cdots,0),$$

i.e. when we don't write an argument of a, this one is equal to zero and having in mind the operator ∂^* introduced by (3), we will write

$$1 + \{u\}\frac{\partial}{\partial x_\ell} := \partial_\ell^*.$$

Now, we are ready to formulate our main result.

Theorem 1. *Let* $\{a_{\mathbf{j}} : \mathbf{j} = (j_1,\cdots,j_d) : 0 \le j_l \le n_l\}$ *be a d-dimensional sequence which one is the restriction of function* $a_{\mathbf{x}} = a(x_1,\cdots,x_d)$ *to* $\mathbf{N}_0^d$ *and assume that* $a(\mathbf{x})$ *satisfies the differentiability condition* (2)*:*

$$\frac{\partial^d a}{\partial x_1\cdots\partial x_d} \in C([0,n_1]\times\cdots\times[0,n_d]).$$

Then

$$\begin{aligned}\sum_{\ell=1}^{d}\sum_{j_\ell=0}^{n_\ell} a_{\mathbf{j}} =& a_0 + \sum_{\ell=1}^{d}\int_0^{n_\ell} \partial_\ell^*\, a(x_\ell)dx_\ell \\ &+ \sum_{1\le j<k\le d}\int_0^{n_j}\int_0^{n_k} \partial_j^*\partial_k^* a(x_j,x_k)dx_jdx_k + \\ &\cdots + \int_0^{n_1}\int_0^{n_2}\cdots\int_0^{n_d} \partial_1^*\partial_2^*\cdots\partial_d^*\, a(x_1,x_2,\cdots,x_d)dx_1dx_2\cdots dx_d.\end{aligned} \tag{6}$$

Proof. The case $d = 2$ is good as any other $d > 2$ to prove the multiple Euler-Maclaurin formula (6). Now, by means of (4) the following conclusions are valid

$$\begin{aligned}\sum_{j=0}^{m}\sum_{k=0}^{n}a_{jk}&=\sum_{j=0}^{m}\left(\sum_{k=0}^{n}a_{jk}\right)=\sum_{j=0}^{m}\left(a_{j0}+\int_0^n\partial_v^*\,a_j(v)dv\right)\\&=\sum_{j=0}^{m}a_{j0}+\int_0^n\left(\sum_{j=0}^{m}a_j(v)+\{v\}\frac{\partial}{\partial v}\sum_{j=0}^{m}a_j(v)\right)dv\\&=a_0+\int_0^m\partial_u^*a(u)du+\int_0^n\left(a(v)+\int_0^m\partial_u^*a(u,v)du\right.\\&\quad\left.+\{v\}\frac{\partial}{\partial v}\left(a(u,v)+\int_0^m\partial_u^*a(u,v)du\right)\right)dv\\&=a_0+\int_0^m\partial_u^*a(u)du+\int_0^n\partial_v^*a(v)dv+\int_0^m\int_0^n\partial_u^*\partial_v^*a(u,v)dudv.\end{aligned}$$

□

This finishes the proof of the assertion.

For the finite onedimensional sum $\sum_{\ell\in\mathbf{I}}a_\ell$, $\mathbf{I}\subset\mathbf{N}$, there exists Euler-Maclaurin summation formula generated by higher order Bernoulli polynomials, requiring higher order differentiability for a_x too, compare [1, Theorem D*, p.95]. But showing here by the formula (6) that with growing number of dimensions the summation formula becomes rather complicated, the higher order Bernoulli polynomials instead of reducing the Euler-Maclaurin summation formula to closed expression trandform it into highly complicated formula with poor future in applications.

For more general cases of multiple Euler-Maclaurin finite summation formula the interested reader can consult the articles [2], [4], [5].

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ПОВЕЌЕКРАТНА ОЈЛЕР-МАКЛОРЕНОВА СУМАЦИОНА ФОРМУЛА

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Резиме

Добиена е d-димензионална сумациона формула за низи кои-што се рестрикции по координати еднаш диференцијабилни ненегативни реални функции.

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