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SOME CLASSES OF L^1 – CONVERGENCE OF FOURIER SERIES

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Abstract

We study here L^1-convergence of a trigonometric series, i.e. the extension is made for the Č. V. Stanojevic and V. B. Stanojevic theorem [5] and also for the theorem proved by the author in[7]. Namely, the new necessary-sufficient conditions for L^1-convergence of Fourier series with δ-quasi-monotone coefficients are obtained.

Then it is verified that the three classes of Fourier coefficients defined by Fomin; Č. V. Stanojevic, V. B. Stanojevic and the author of the present note, are identical.

1. Introduction

It is well known that if a trigonometric series converges in L^1 to a function $f \in L^1$ then it is the Fourier series of the function f. Let $\{c_k: k = 0, \pm 1, \pm 2, \ldots\}$ be a sequence of complex numbers, and the partial sums of the complex trigonometric series

$$\sum_{|n| \leq \infty} c_n e^{int} \tag{1.1}$$

be denoted by $S_n(c,t) = \sum_{k=-n}^{n} c_k e^{ikt}$, $t \in T$, where T is the unit circle.

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If the trigonometric series is the Fourier series of some $f \in L^1$, we shall write $c_n = \hat{f}(n)$, for all n and $S_n(c,t) = S_n(f,t) = S_n(f)$. Let $E_n(t) = \sum_{k=0}^{n} e^{ikt}$.

The complex form of the above modified trigonometric sums is

$$g_n(c,t) = S_n(c,t) - \big(c_n E_n(t) + c_{-n} E_{-n}(t)\big).$$

We assume that $\{c_n\}$ is a null sequence i.e.

$$\lim_{|n|\to\infty} c_n = 0. \tag{1.2}$$

A complex null sequence $\{c_n\}$ satisfyng

$$\sum_{n=1}^{\infty} |\Delta(c_n - c_{-n})| \log n < \infty \tag{1.3}$$

is called weakly even. It is obvious that if $\{c_n\}$ is an even sequence then it is weakly even. Č. V. Stanojevic and V. B. Stanojevic introduced a class S_p^* defined as follows: Aweakly even null sequence $\{c_n\}$ of complex numbers belongs to the class S_p^* if for some $1 < p \le 2$ and some monotone sequence $\{A_n\}$ such that $\sum_{n=1}^{\infty} A_n < \infty$ the condition $\frac{1}{n}\sum_{k=1}^{n} \frac{|\Delta c_k|^p}{A_k^p} = O(1)$ holds.

They [5] proved the following theorem:

Theorem A. *Let* $\{c_n\} \in S_p^*$, $1 < p \le 2$. *Then*

(i) *for* $t \neq 0$, $\lim_{n\to\infty} S_n(C,t) = f(t)$ *exists.*

(ii) $f \in L^1(T)$

(iii) $||S_n(f) - f|| = o(1)$, $n \to \infty$ *is equivalent to* $\hat{f}(n)\log|n| = o(1)$, $n \to \infty$.

On the other hand, a sequence $\{A_k\}$ is said to be δ-quasi-monotone if $A_k \to 0$, $A_k > 0$ ultimately and $\Delta A_k \ge -\delta_k$, where $\{\delta_k\}$ is a sequence of positive numbers.

We introduce here a new class $S_{p_\alpha}^*(\delta)$, $1 < p \le 2$, $\alpha \ge 0$ of sequence as follows. A weakly even sequence $\{c_k\}$ of complex numbers satisfies conditions $S_{p\alpha}^*(\delta)$, or $c_k \in S_{p\alpha}^*(\delta)$ if $c_k \to 0$ as $k \to \infty$, and there exists a sequence of numbers $\{A_k\}$ such that:

a) $\{A_k\}$ is δ-quasi-monotone and $\sum_{k=1}^{\infty} k^{1+\alpha}\delta_k < \infty$

b) $\sum_{k=1}^{\infty} k^{\alpha} A_k < \infty$ c) $\frac{1}{n^{p\alpha+1}} \sum_{k=1}^{n} \frac{|\Delta c_k|^p}{A_k^p} = O(1)$.

Let $\{c_n\}$ is an even and real sequence ($c_{-k} = c_k = a_k$, for all k). Then series (1.1) is a cosine series:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx} = 2\left(\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx\right). \tag{C}$$

In this case, we denote $S_p^* = S_p$. But in [6], we defined a new integrability class $S_p(\delta)$, $p > 1$ (case $\alpha = 0$). Later the author of the present note proved the following theorem.

Theorem B [8] *For any $p > 1$, the classes S_p and $S_p(\delta)$ are identical.*

Fomin [3] have defined a class $\mathcal{F}_p$, $p > 1$ of Fourier coefficients as follows: a sequence $\{a_k\}$ belongs to $\mathcal{F}_p$, $p > 1$ if $a_k \to 0$ as $k \to \infty$ and

$$\sum_{k=1}^{\infty} \left(\frac{1}{k} \sum_{i=k}^{\infty} |\Delta a_i|^p\right)^{1/p} < \infty.$$

Theorem C [3]. *Let $\{a_n\} \in \mathcal{F}_p$, $1 < p \leq 2$. Then the cosine series (C) is the Fourier series of its sumf and $||S_n - f|| = o(1)$, $n \to \infty$ iff $a_n \log n = o(1)$, $n \to \infty$.*

In [4] L. Leindler proved the following very important result.

Theorem D [4] *For any $p > 1$ the classes S_p and F_p are identical.*

2. Lemmas

For the proofs of our results, we need the following Lemmas.

Lemma 1 [2]. *For each non-negative integer n, there holds $||\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t)|| = o(1)$, $n \to \infty$ if and only if $\hat{f}(n) \log |n| = o(1) \to \infty$.*

The folloowing lemma, was proved by Boas in [1], but now we shall give a new elegant proof different from that of Boas.

Lemma 2 [1]. If $\{A_n\}$ is a δ-quasi-monotone sequence with $\sum_{n=1}^{\infty} n^{1+\alpha}\delta_n < \infty$, $\alpha \geq 0$, then the convergence of $\sum_{n=1}^{\infty} n^{\alpha} A_n < \infty$ implies that $n^{1+\alpha} A_n = o(1)$, $n \to \infty$.

Proof. By inequalities

$$n^{\alpha+1}A_{2n} \le n^{\alpha}\Big[\Big(A_n + \sum_{k=n}^{2n-1}\delta_k\Big) + \Big(A_{n+1} + \sum_{k=n+1}^{2n-1}\delta_k\Big) +$$
$$+ \cdots + (A_{2n-1} + \delta_{2n-1})\Big] \le n^{\alpha}\left(\sum_{i=n}^{2n-1} A_i + \sum_{i=n}^{2n-1} i\delta_i\right)$$

we obtain

$$(2n)^{\alpha+1}A_{2n} \le 2^{\alpha+1}\sum_{i=n}^{\infty} i^{\alpha}A_i + 2^{\alpha+1}\sum_{i=n}^{\infty} i^{\alpha+1}\delta_i =$$
$$= o(1) + o(1) = o(1)\,, \quad n \to \infty\,,$$

$$(2n+1)^{\alpha+1}A_{2n+1} \le (2n)^{\alpha+1}\Big(1 + \frac{1}{2n}\Big)^{\alpha+1}(A_{2n} + \delta_{2n}) \le$$
$$\Big(\frac{3}{2}\Big)^{\alpha+1}\Big[(2n)^{\alpha+1}A_{2n} + (2n)^{\alpha+1}\delta_{2n}\Big] = o(1)\,, \quad n \to \infty\,.$$

Hence $n^{\alpha+1}A_n = o(1)$, $n \to \infty$.

Lemma 3 [1]. *Let* $\{A_n\}$ *is a* δ*-quasi-monotone sequence with* $\sum_{n=1}^{\infty} n^{1+\alpha}\delta_n < \infty$, $\alpha \ge 0$. *If* $\sum_{n=1}^{\infty} n^{\alpha}A_n < \infty$ *then* $\sum_{n=1}^{\infty} n^{1+\alpha}|\Delta A_n| < \infty$.

Lemma 4 [3]. *A sequence* $\{a_n\} \in \mathcal{F}_p$, $p > 1$ *iff* $\sum_{s=1}^{\infty} 2^s \Delta s^{(p)} < \infty$ *where*

$$\Delta s^{(p)} = \left\{\frac{1}{2^{s-1}} \sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p\right\}^{1/p}.$$

Lemma 5 *If* $\{c_n\} \in S^*_{p\alpha}(\delta)$, $1 < p \le 2$, $\alpha \ge 0$, *then following relation holds,*

$$\int_0^{\pi}\left|\sum_{j=1}^{k} \frac{\Delta c_j}{A_j} D_j(t)\right| dt = O_p(k^{\alpha+1})\,,$$

where O_p *depends only on* p.

Proof. We have

$$\int_0^{\pi}\left|\sum_{j=1}^{k} \frac{\Delta c_j}{A_j} D_j(t)\right| dt = \int_0^{\pi/k} + \int_{\pi/k}^{\pi} = I_k + J_k\,.$$

Applying the Holder-Hausdorff-Young technique as in the proof of the theorems of [6] and [7], we have:

$$I_k \leq Mk\left(\frac{1}{k}\sum_{j=1}^{k}\frac{|\Delta c_k|^p}{A_k^p}\right)^{\frac{1}{p}} = Mk^{\alpha+1}\left(\frac{1}{k^{p\alpha+1}}\sum_{j=1}^{k}\frac{|\Delta c_j|^p}{A_j^p}\right)^{\frac{1}{p}} = O(k^{\alpha+1}),$$

where $M > 0$ and

$$J_k \leq \left(\frac{\pi}{p-1}\right)^{\frac{1}{p}} k^{(p-1)/p}\left(\sum_{j=1}^{k}\frac{|\Delta c_j|^p}{A_j^p}\right)^{\frac{1}{p}}$$

$$= D_p k^{\alpha+1}\left(\frac{1}{k^{p\alpha+1}}\sum_{j=1}^{k}\frac{|\Delta c_j|^p}{A_j^p}\right)^{\frac{1}{p}} = O_p(k^{\alpha+1}),$$

where $D_p > 0$.Thus

$$\int_0^\pi \left|\sum_{j=1}^{k}\frac{\Delta c_j}{A_j}D_j(t)\right| dt = O(k^{\alpha+1}) + O_p(k^{\alpha+1}) = O_p(k^{\alpha+1}).$$

Lemma 6 *If* $\{c_n\} \in S^*_{p\alpha}(\delta)$, $1 < p \leq 2$, $\alpha \geq 0$, *then*

$$A_n \int_0^\pi \left|\sum_{j=1}^{n}\frac{\Delta c_j}{A_j}D_j(t)\right| dt = o(1), \quad n \to \infty$$

Proof. Applying firstly Lemma 5, then Lemma 2, we obtain:

$$A_n \int_0^\pi \left|\sum_{j=1}^{n}\frac{\Delta c_j}{A_j}D_j(t)\right| dt = O_p(n^{\alpha+1}A_n) = o(1), \quad n \to \infty.$$

3. Main results

We shall prove the following results.

Theorem 1. *Let* $\{c_n\} \in S^*_{p\alpha}(\delta)$, $1 < p \leq 2$. *Then,*

(i) *for* $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ *exists.*

(ii) $f \in L^1(T)$

(iii) $\|S_n(f) - f\| = o(1)$, $n \to \infty$ *is equivalent to* $\hat{f}(n) \log|n| = o(1)$, $n \to \infty$.

Proof. Applying the same technique as in proof of the theorem of [6], we have:

$$\sum_{k=1}^{n} |\Delta c_k| \le \sum_{k=1}^{n-1} k|\Delta A_k| \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta c_j|^p}{A_j^p} \right)^{\frac{1}{p}} + nA_n \left(\frac{1}{n} \sum_{j=1}^{n} \frac{|\Delta c_j|^p}{A_j^p} \right)^{\frac{1}{p}}$$

$$= \sum_{k=1}^{n-1} k^{1+\alpha} |\Delta A_k| \left(\frac{1}{k^{p\alpha+1}} \sum_{j=1}^{k} \frac{|\Delta c_j|^p}{A_j^p} \right)^{\frac{1}{p}} + n^{1+\alpha} A_n \left(\frac{1}{n^{p\alpha+1}} \sum_{j=1}^{n} \frac{|\Delta c_j|^p}{A_j^p} \right)^{\frac{1}{p}}$$

$$= O(1) \left(\sum_{k=1}^{n-1} k^{1+\alpha} |\Delta A_k| + n^{\alpha+1} A_n \right).$$

Hence $\{c_n\}$ is of bounded variation and for $t \neq 0$, $\lim_{n\to\infty} S_n(c,t) = f(t)$ exists. Then,

$$g_n(c,t) = \sum_{k=1}^{n-1} \left(\Delta(c_{-n} - c_k)\right)\left(E_{-k}(t) - 1\right) - c_{-n} + \sum_{k=0}^{n} \Delta c_k D_k(t).$$

From (i) it follows that for $t \neq 0$,

$$f(t) - g_n(c,t) = \sum_{k=n}^{\infty} \Delta c_k D_k(t) + \sum_{k=n}^{\infty} \Delta(c_{-k} - c_k) E_{-k}(t).$$

From the last identity we have the estimate:

$$\|f - g_n(c)\| \le \int_T \left| \sum_{k=n}^{\infty} \Delta c_k D_k(t) \right| dt + A \sum_{k=n}^{\infty} |\Delta(c_{-k} - c_k)| \log k,$$

where A is an absolute constant. Thus

$$\|f - g_n(c)\| \le B \int_0^{\pi} \left| \sum_{k=n}^{\infty} \Delta c_k D_k(t) \right| dt + o(1), \quad n \to \infty,$$

where B is an absolute constant. Then applying the Abels' transformation, Lemma 5 and Lemma 6, we obtain:

$$\int_0^\pi \left|\sum_{k=n}^{\infty} \Delta c_k D_k(t)\right| dt \le \sum_{k=n}^{\infty} |\Delta A_k| \int_0^\pi \left|\sum_{j=1}^{k} \frac{\Delta c_j}{A_j} D_j(t)\right| dt$$

$$+A_n \int_0^\pi \left|\sum_{j=1}^{n-1} \frac{\Delta c_j}{A_j} D_j(t)\right| dt = O_p(1) \left(\sum_{k=n}^{\infty} |\Delta A_k| k^{\alpha+1} + n^{\alpha+1} A_n\right).$$

Then by Lemma 2 and Lemma 3, we get $\|f - g_n(c)\| = o(1)$, $n \to \infty$. Since g_n is a polynomial, it follows that $f \in L^1(T)$.

The proof (iii) follows from the estimate:

$$\left| \|f - S_n(f)\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| \le \|f - g_n(c)\| = o(1)\,, \quad n \to \infty\,,$$

and from the Lemma 1.

Corollary 1 [7]. *Let $\{a_n\} \in S_p(\delta)$, $1 < p \le 2$. Then cosine series* (C) *is the Fourier series of its sum f and $\|S_n - f\| = o(1)$, $n \to \infty$ iff $a_n \log n = o(1)$, $n \to \infty$.*

In [3], Fomin note that it is easy to see that the class $\mathcal{F}_p$ is wider when p is closer to 1. But now we shall present the proof of this fact.

Theorem 2. *For any $1 < r < p$ the following embedding relation holds: $\mathcal{F}_p \subset \mathcal{F}_r$.*

Proof. By inequality $\frac{1}{r} > \frac{1}{p}$, we have $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$, where $q > 0$. This equality imply that $\frac{1}{p'} + \frac{1}{q'} = 1$, where $p' = \frac{p}{r}$ and $q' = \frac{q}{r}$. Applying the Hölder's inequality, we have:

$$\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^r = \sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^r \cdot 1 \le \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^{rp'}\right)^{\frac{1}{p'}} \left(\sum_{k=2^s+1}^{2^{s+1}} 1^{q'}\right)^{\frac{1}{q'}}$$

$$= (2^s)^{1/q'} \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^p\right)^{\frac{1}{p'}}$$

Then

$$\sum_{s=1}^{\infty} 2^s \Delta s^{(r)} \le \sum_{s=1}^{\infty} 2^s \cdot 2^{-s/r} \cdot 2^{s/q'r} \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^p \right)^{\frac{1}{p}\frac{1}{r}}$$

$$= \sum_{s=1}^{\infty} 2^s \left(\frac{1}{2^s}\right)^{\frac{1}{r}-\frac{1}{q}} \left(\sum_{k=2^s+1}^{2^{s+1}} |\Delta a_k|^p \right)^{\frac{1}{p}} = \sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} .$$

Applying the Lemma 4, the proof of the theorem is completed.

Theorem 3. *For any $p > 1$ the classes S_p, $S_p(\delta)$ and $\mathcal{F}_p$ are identical.*

Proof. The proof follows from Theorem B and Theorem D.

Remark. *Concerning the embedding relation $S_p \subseteq \mathcal{F}_p$, L. Leindler* [4] *verified the inequalities*

$$\sum_{m=1}^{\infty} 2^{m(1-\frac{1}{p})} \left\{ \sum_{n=2^m+1}^{2^{m+1}} |\Delta a_n|^p \right\}^{\frac{1}{p}} \le \sum_{m=1}^{\infty} 2^m A_{2^m} \left\{ 2^{-m} \sum_{n=2^m+1}^{2^{m+1}} \frac{|\Delta a_n|^p}{A_n^p} \right\}^{\frac{1}{p}} < \infty$$

Now we shall present a new proof of this embedding relation, different from that of L. Leindler.

Namelly, applying the Abel's transformation, we have:

$$\sum_{k=2^{s-1}+1}^{2^s} |\Delta a_k|^p = \sum_{k=2^{s-1}+1}^{2^s} A_k^p \frac{|\Delta a_k|^p}{A_k^p}$$

$$= \sum_{k=2^{s-1}+1}^{2^s-1} \Delta(A_k^p) \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} + A_{2^s}^p \sum_{j=1}^{2^s} \frac{|\Delta a_j|^p}{A_j^p} - A_{2^{s-1}+1}^p \sum_{j=1}^{2^{s-1}} \frac{|\Delta a_j|^p}{A_j^p}$$

$$= \sum_{k=2^{s-1}+1}^{2^s-1} k\,\Delta(A_k^p) \left(\frac{1}{k} \sum_{j=1}^{k} \frac{|\Delta a_j|^p}{A_j^p} \right) + 2^s A_{2^s}^p \left(\frac{1}{2^s} \sum_{j=1}^{2^s} \frac{|\Delta a_j|^p}{A_j^p} \right)$$

$$- 2^{s-1} A_{2^{s-1}+1}^p \left(\frac{1}{2^{s-1}} \sum_{j=1}^{2^{s-1}} \frac{|\Delta a_j|^p}{A_j^p} \right)$$

$$= O(1) \left[\sum_{k=2^{s-1}+1}^{2^s-1} k\,\Delta(A_k^p) + 2^s A_{2^s}^p + 2^{s-1} A_{2^{s-1}+1}^p \right]$$

$$=O(1)\left[\sum_{k=2^{s-1}+1}^{2^s} A_k^p - 2^s A_{2^s}^p + 2^s A_{2^s}^p + 2^{s-1} A_{2^{s-1}+1}^p\right]$$
$$=O(2^{s-1} A_{2^{s-1}}^p).$$

Applying the Cauchy condensation test, we obtain

$$\sum_{s=1}^{\infty} 2^s \Delta_s^{(p)} \le O(1) \sum_{s=1}^{\infty} 2^s \left(\frac{1}{2^{s-1}} 2^{s-1} A_{2^{s-1}}^p\right)^{\frac{1}{p}}$$
$$= O\left(\sum_{s=1}^{\infty} 2^{s-1} A_{2^{s-1}}\right) < \infty .$$

Finally, by Lemma 4, *we obtain* $\{a_k\} \in \mathcal{F}_p$.

References

[1] R. P. Boas: *Quasi-positive sequence and trigonometric series*, Proc. London Math. Soc. **14A** (1965), 38-48.

[2] W. O. Bray and Č. V. Stanojevic: *L^1-convergence classes of Fourier series*, Transs. Amer. Math. Soc. **275** (1983), 59-69.

[3] G. A. Fomin: *A class of trigonometric series*, Mat. Zametki, **23** (1978), 117-124. (Russian)

[4] L. Leindler: *On the equivalence of classes of Fourier coefficients*, Math. Inequalities & Applications, **3** (1) (2000), 45-50.

[5] Č. V. Stanojevic and V. B. Stanojević: *Generalizations of the Sidon-Telyakovskii theorem*, Proc. Amer. Math. Soc. **101** (1987), 679-684.

[6] Ž. Tomovski, *An application of the Hausdorff-Young inequality*, Math. Inequalities & Appl., **1** (4) (1998), 527-532.

[7] Ž. Tomovski, *Necessary and sufficient condition for L^1 convergence of cosine trigonometric series with δ-quasi monotone coefficients*, Mathematical Communications, (**4**) N.2 (1999), 219-224.

[8] Ž. Tomovski, *Remarks on some classes of Fourier coefficients, Analysis Mathematica*, (Szeged-Moscow), (accepted).

НЕКОИ КЛАСИ НА L^1-КОНВЕРГЕНЦИЈА НА ФУРИЕОВ РЕД

Томовски Живорад

Резиме

Во трудот се разгледува проблемот на L^1-конвергенција на тригонометриски ред, т.е. дадено е проширување на теоремата на Ч. В. Станојевиќ и В. Б. Станојевиќ [5], а исто така и на теоремата од авторот докажана во [7]. Имено, дадени се потребни и доволни услови за L^1-конвергенција на Фуриеов ред со δ-квазимонотони коиефициенти. Потоа е утврдено дека трите класи Фуриеови коефициенти, дефинирани од Фомин; Ч. В. Станојевиќ, В. Б. Станојевиќ [5] и авторот од овој труд се идентични.

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