LINEARIZATION OF A PRODUCT OF TWO POLYNOMIALS OF DIFFERENT ORTHOGONAL SYSTEMS

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Abstract. In the present note, we deal with the problem of linearization of a product of two polynomials of different orthogonal systems. In fact, we give the expansions of a product of Laguerre and Legendre polynomials in series of such polynomials. As an application of our main expansion formulas, we evaluate some definite integrals of products of the considered polynomials.

1. Introduction and Preliminaries

We start with a classical technique for expanding the Laguerre polynomials:

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}$$

in a series of the Legendre polynomials given by

$$P_n(2x-1) := \sum_{k=0}^{n} (-1)^{n+k} \frac{(n+k)!}{(n-k)!} \frac{x^k}{(k!)^2},$$

for which we have (cf. [3, p. 185, Exercise 17])

$$x^n = (n!)^2 \sum_{k=0}^{n} \frac{2k+1}{(n-k)!\,(n+k+1)!}\, P_k\,(2x-1)\;.$$

See, for details, [3], [4] and [5].

Consider the series

$$\begin{aligned} G(x,t) &= \sum_{n=0}^{+\infty} L_n^{(\alpha)}(x) t^n = \sum_{n=0}^{+\infty}\sum_{s=0}^{n} \frac{(-1)^s(1+\alpha)_n}{(1+\alpha)_s} \frac{t^n}{(n-s)!} \frac{x^s}{s!} \\ &= \sum_{n,s=0}^{+\infty} \frac{(-1)^s(1+\alpha)_{n+s}}{(1+\alpha)_s} \frac{t^n}{n!} \frac{(xt)^s}{s!}, \end{aligned}$$

i.e.,

$$\begin{aligned} G(x,t) &= \sum_{n,s=0}^{+\infty}\sum_{k=0}^{s} \frac{(-1)^s(1+\alpha)_{n+s}\, s!}{n!(1+\alpha)_s(s+k+1)!(s-k)!}\, (2k+1)\, P_k(2x-1)t^{n+s} \\ &= \sum_{n,k,s=0}^{+\infty} \frac{(-1)^{s+k}(1+\alpha)_{n+s+k}(s+k)!}{n!(1+\alpha)_{s+k}(s+2k+1)!\; s!}\, (2k+1)P_k(2x-1)t^{n+s+k} \\ &= \sum_{n,k=0}^{+\infty}\sum_{s=0}^{n} \frac{(-1)^k(-n)_s(k+1)_s k!(1+\alpha)_{n+k}}{n!(1+\alpha+k)_s s!(2k+2)_s(2k)!(1+\alpha)_k}\, P_k(2x-1)t^{n+k} \\ &= \sum_{n,k=0}^{+\infty} {}_2F_2\left(\begin{matrix} -n, k+1; \\ 1+\alpha+k, 2k+2; \end{matrix} 1\right) \frac{(-1)^k k!(1+\alpha)_{n+k}}{(2k)!n!(1+\alpha)_k} P_k(2x-1)t^{n+k} \\ &= \sum_{n=0}^{+\infty}\sum_{k=0}^{n} {}_2F_2\left(\begin{matrix} -n+k, k+1; \\ 1+\alpha+k, 2+2k; \end{matrix} 1\right) \frac{(-1)^k k!(1+\alpha)_n}{(n-k)!(2k)!(1+\alpha)_k} P_k(2x-1)t^n. \end{aligned}$$

We thus conclude that (cf. [1, p. 151, Eq. (4.3) with $a = b = 1$ and $\lambda = \mu = 0$])

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_2F_2\left(\begin{matrix} -n+k, k+1; \\ 1+\alpha+k,\ 2+2k; \end{matrix} 1\right) \frac{(-1)^k k!(1+\alpha)_n}{(n-k)!(2k)!(1+\alpha)_k} P_k(2x-1). \tag{1.1}$$

Here, and throughout this presentation, ${}_pF_q$ denotes a generalized hypergeometric function with p numerator and q denominator parameters and $(\lambda)_\nu$ denotes the Pochhammer symbol or the shifted factorial, since

$$(1)_n = n! \ (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\,;\ \mathbb{N} := \{1, 2, 3, \ldots\}),$$

defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C}\setminus\{0\}), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

The main object of this presentation is to investigate the problem of linearization of a product of the Laguerre and Legendre polynomials in terms of each of these polynomials. We also apply the expansion formulas derived here in evaluating some definite integrals involving products of the Laguerre and Legendre polynomials.

2. Consequences of the Neumann-Adams Formula

The Neumann-Adams formula gives the expansion of a product of two Legendre polynomials in a series of such polynomials as follows [6, p. 331, Example 11]:

$$\begin{aligned} P_m(x)P_n(x) &= \sum_{r=0}^{m} \frac{A_{m-r}A_rA_{n-r}}{A_{m+n-r}} \left(\frac{2m+2n-4r+1}{2m+2n-2r+1}\right) P_{m+n-2r}(x) \\ &= \sum_{r=0}^{m} A_{m,n}^r\, P_{m+n-2r}(x), \end{aligned} \tag{2.1}$$

where, for convenience,

$$A_{m,n}^r := \frac{A_{m-r}A_rA_{n-r}}{A_{m+n-r}} \left(\frac{2m+2n-4r+1}{2m+2n-2r+1}\right)$$

and

$$A_m := \frac{1\cdot 3\cdot 5\cdots(2m-1)}{m!} \qquad (m, n \in \mathbb{N};\ n \geq m > 1)\ .$$

Now, rewriting (1.1) in the form:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \Theta_{n,k} P_k(2x-1)$$

with

$$\Theta_{n,k} := {}_2F_2\left(\begin{matrix} -n+k, k+1; \\ 1+\alpha+k,\ 2+2k; \end{matrix} 1\right) \frac{(-1)^k k!(1+\alpha)_n}{(n-k)!(2k)!(1+\alpha)_k},$$

we find from the Neumann-Adams formula (2.1) that

$$
\text{(2.2)}\quad
\begin{aligned}
L_n^{(\alpha)}(x)P_m(2x-1) &= \sum_{k=0}^{n} \Theta_{n,k}\ P_k(2x-1)P_m(2x-1) \\
&= \sum_{k=0}^{n}\sum_{r=0}^{k} \Theta_{n,k}\ A_{k,m}^{r}\ P_{k+m-2r}(2x-1),
\end{aligned}
$$

which obviously linearizes the product of the Laguerre and Legendre polynomials in terms of the Legendre polynomials.

By using yet another expansion of the Laguerre polynomials in a series of the Legendre polynomials [3, p. 216, Exercise 2]:

$$
L_n^{(\alpha)}(x) = \sum_{k=0}^{n} {}_2F_3\left(\begin{matrix} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1); \\ \frac{3}{2}+k,\ \frac{1}{2}(1+\alpha+k),\ \frac{1}{2}(2+\alpha+k); \end{matrix}\ \frac{1}{4}\right) \times
$$

$$
\times \frac{(-1)^k(1+\alpha)_n}{2^k(n-k)!\left(\frac{3}{2}\right)_k(1+\alpha)_k}\ (2k+1)P_k(x) = \sum_{k=0}^{n} \Phi_{n,k}\ P_k(x),
$$

where

$$
\Phi_{n,k} := {}_2F_3\left(\begin{matrix} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1); \\ \frac{3}{2}+k, \frac{1}{2}(1+\alpha+k), \frac{1}{2}(2+\alpha+k); \end{matrix}\ \frac{1}{4}\right) \frac{(-1)^k(2k+1)(1+\alpha)_n}{2^k(n-k)!\left(\frac{3}{2}\right)_k(1+\alpha)_k},
$$

we can similarly express the product of the Laguerre and Legendre polynomials in the following linearization formula:

$$
\text{(2.3)}\qquad L_n^{(\alpha)}(x)P_m(x) = \sum_{k=0}^{n}\sum_{r=0}^{k} \Phi_{n,k}\ A_{k,m}^{r}\ P_{k+m-2r}(x).
$$

3. Consequences of the Feldheim Formula

For the expansion of a Legendre polynomial $P_n(2x-1)$ in a series of the Laguerre polynomials, we know that (cf. [1, p. 150, Equation (4.2) with $a=b=1$, $\alpha=\beta=0$, and $\lambda \to \alpha$])

$$
\text{(3.1)}\quad P_n(2x-1) = \sum_{k=0}^{n} {}_3F_1\left(\begin{matrix} -n+k,\ 1+n+k,\ 1+\alpha+k; \\ 1+k; \end{matrix}\ 1\right) \times
$$

$$
\times \frac{(-1)^k(n+k)!}{(n-k)!\,k!}\ L_k^{(\alpha)}(x) = \sum_{k=0}^{n} \Psi_{n,k}\ L_k^{(\alpha)}(x),
$$

where

$$
\Psi_{n,k} := {}_3F_1\left(\begin{matrix} -n+k,\ 1+n+k, 1+\alpha+k; \\ 1+k; \end{matrix}\ 1\right) \frac{(-1)^k(n+k)!}{(n-k)!\ k!}.
$$

Now the Feldheim formula [2], which expresses the product of two Laguerre polynomials as a sum of Laguerre polynomials, is given by

$$
\text{(3.2)}\quad
\begin{aligned}
L_m^{(\alpha)}(x)L_n^{(\beta)}(x) &= \sum_{s=0}^{m+n} C_s(m,n,\alpha,\beta)L_s^{(\alpha+\beta)}(x) \\
&= (-1)^{m+n}\sum_{s=0}^{m+n} C_s(m,n,\beta-m+n,\alpha+m-n)\frac{x^s}{s!},
\end{aligned}
$$

with

$$C_s(m,n,\alpha,\beta) := (-1)^{m+n+s} \sum_{r=0}^{s} \binom{s}{r} \binom{m+\alpha}{n-s+r} \binom{n+\beta}{m-r}$$

$$(\Re(\alpha) > -1;\ \Re(\beta) > -1;\ \Re(\alpha+\beta) > -1).$$

Thus, by making use of (3.1) in conjunction with the Feldheim formula (3.2), we obtain

$$\begin{aligned} L_m^{(\beta)}(x) P_n(2x-1) &= \sum_{k=0}^{n} \Psi_{n,k}\, L_k^{(\alpha)}(x) L_m^{(\beta)}(x) \\ &= \sum_{k=0}^{n} \sum_{s=0}^{m+\kappa} \Psi_{n,k}\, C_s(k,m,\alpha,\beta) L_s^{(\alpha+\beta)}(x) \\ &= \sum_{k=0}^{n} \sum_{s=0}^{m+k} (-1)^{m+k} \Psi_{n,k}\, C_s(k,m,\beta-k+m,\alpha+k-m) \frac{x^s}{s!}. \end{aligned}$$

Another expansion of $P_n(x)$ in a series of the Laguerre polynomials is given by [3, p. 208, Equation (4)]

$$P_n(x) = \frac{2^n \left(\frac{1}{2}\right)_n (1+\alpha)_n}{n!} \times$$

$$\times \sum_{k=0}^{n} {}_2F_3\left(\begin{matrix} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1); \\ \frac{1}{2}-n, -\frac{1}{2}(\alpha+n), -\frac{1}{2}(\alpha+n-1); \end{matrix} \frac{1}{4} \right) \frac{(-n)_k}{(1+\alpha)_k} L_k^{(\alpha)}(x)$$

$$= \sum_{k=0}^{n} \Xi_{n,k}\, L_k^{(\alpha)}(x), \tag{3.3}$$

where

$$\Xi_{n,k} := \frac{2^n \left(\frac{1}{2}\right)_n (1+\alpha)_n (-n)_k}{n!\,(1+a)_k} {}_2F_3\left(\begin{matrix} -\frac{1}{2}(n-k), -\frac{1}{2}(n-k-1); \\ \frac{1}{2}-n, -\frac{1}{2}(\alpha+n), -\frac{1}{2}(\alpha+n-1); \end{matrix} \frac{1}{4} \right).$$

By virtue of (3.2) and (3.3), for the product of a Legendre polynomial and a Laguerre polynomial, we have

$$P_n(x) L_m^{(\beta)}(x) = \sum_{k=0}^{n} \Xi_{n,k}\, L_k^{(\alpha)}(x) L_m^{(\beta)}(x) = \sum_{k=0}^{n} \sum_{s=0}^{k+m} \Xi_{n.k}\, C_s(k,m,\alpha,\beta) L_s^{(\alpha+\beta)}(x).$$

4. Applications of the Linearization Formulas

For the Legendre polynomials, the following orthogonality property is well-known:

$$\int_{-1}^{1} P_n(x) P_m(x)\,dx = \frac{2}{2n+1}\,\delta_{m,n} \tag{4.1}$$

or, equivalently,

$$\int_{0}^{1} P_n(2x-1) P_m(2x-1)\,dx = \frac{1}{2n+1}\,\delta_{m,n} \tag{4.2}$$

where $\delta_{m,n}$ denotes the Kronecker delta.

Making use of (4.2), the linearization relation (2.2) leads us to the following integral formula:

$$\int_0^1 L_n^{(\alpha)}(x)P_m(2x-1)dx=\sum_{k=0}^{n}\Theta_{n,k}\int_0^1 P_k(2x-1)P_m(2x-1)dx=\frac{\Theta_{n,m}}{2m+1}$$

$$=\frac{(-1)^m m!(1+\alpha)_n}{(n-m)!(2m+1)!(1+\alpha)_m}\,{}_2F_2\left(\begin{matrix}m-n,\ 1+m;\\ 1+\alpha+m,\ 2+2m;\end{matrix}\ 1\right).$$

Similarly, we find from (2.3) and (4.1) that

$$\int_{-1}^1 L_n^{(\alpha)}(x)P_m(x)\,dx=\frac{2}{2m+1}\Phi_{n,m}=\frac{(-1)^m(1+\alpha)_n}{2^{m-1}(n-m)!\left(\frac{3}{2}\right)_m(1+\alpha)_m}\times$$

$$\times{}_2F_3\left(\begin{matrix}-\frac{1}{2}(n-m),-\frac{1}{2}(n-m-1);\\ \frac{3}{2}+m,\frac{1}{2}(1+\alpha+m),\frac{1}{2}(2+\alpha+m);\end{matrix}\ \frac{1}{4}\right).$$

In precisely the same manner as indicated above, the well-known orthogonality property:

$$\int_0^{+\infty}x^\alpha e^{-x}L_m^{(\alpha)}(x)L_n^{(\alpha)}(x)dx=\frac{\Gamma(1+\alpha+n)}{n!}\delta_{m,n}\quad(\Re(\alpha)>-1),$$

in conjunction with (3.1) and (3.3), would lead us easily to the integral formulas:

$$\int_0^{+\infty}e^{-x}x^\alpha L_n^{(\alpha)}(x)P_m(2x-1)dx=\sum_{k=0}^{n}\Psi_{n,k}\int_0^{+\infty}e^{-x}x^\alpha L_k^{(\alpha)}(x)L_m^{(\alpha)}(x)dx$$

$$=\frac{\Gamma(1+\alpha+m)}{m!}\Psi_{n,m}=\frac{(-1)^m(m+n)!\Gamma(1+\alpha+m)}{(n-m)!\,(m!)^2}\times$$

$$\times\,{}_3F_1\left(\begin{matrix}m-n,\ 1+m+n,\ 1+\alpha+m;\\ 1+m;\end{matrix}\ 1\right)$$

and

$$\int_0^{+\infty}e^{-x}x^\alpha P_n(x)\,L_m^{(\alpha)}(x)\,dx=\sum_{k=0}^{n}\Xi_{n,k}\int_0^{+\infty}e^{-x}x^\alpha L_k^{(\alpha)}(x)\,L_m^{(\alpha)}(x)\,dx$$

$$=\frac{\Gamma(1+\alpha+m)}{m!}\,\Xi_{n,m}=\frac{(-n)_m\ 2^n\left(\frac{1}{2}\right)_n\,\Gamma(1+\alpha+n)}{(n!)^2}\times$$

$$\times\,{}_2F_3\left(\begin{matrix}-\frac{1}{2}(n-m),\ -\frac{1}{2}(n-m-1);\\ \frac{1}{2}-n,\ \frac{1}{2}(\alpha+n),\ \frac{1}{2}(\alpha+n-1);\end{matrix}\ \frac{1}{4}\right),$$

respectively.

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