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EXTENSION OF AN INTERPOLATION OF JENSEN'S INEQUALITY

Josip E. Pečarić

Abstract. Extension of interpolation of Jensen's inequality for convex functions given by S.S.Dragomir is obtained.

Let $f: C \subseteq X \to R$ is a convex mapping on convex set C of real linear space X, x_i are in C $(i=1,\dots,n)$ and $p_i \geq 0$ with $P_n := \sum_{i=1}^{n} p_i > 0$. Then the well-known Jensen inequality

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \tag{1}$$

is valid.

Let $f_{n,k}$ be defined by

$$f_{n,k} = \frac{1}{P_n^k}\sum_{i_1=1}^{n}\dots\sum_{i_k=1}^{n} p_{i_1}\dots p_{i_k} f\Big(x_{i_1}(1-t_1)+\sum_{j=1}^{k-1} x_{i_j}(1-t_{j+1})t_1\dots t_j + \bar{x}\, t_1\dots t_k\Big),$$

where $\bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i$, and $t_i \in [0,1]$, $i=1,\dots,n-1$.

S.S.Dragomir [1] gave the following interpolation of (1):

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq f_{n,1} \leq \frac{1}{P_n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} p_i p_j f(tx_i+(1-t)x_j) \leq \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{2}$$

where $t_1 = t \in [0,1]$.

Here, we shall note that the following extension of (2) is valid:

$$f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big) \leq f_{n,1} \leq f_{n,2} \leq \dots \leq f_{n,n-1} \leq$$
$$\leq \frac{1}{P_n^n}\sum_{i_1=1}^{n}\dots\sum_{i_n=1}^{n} p_{i_1}\dots p_{i_n} f\Big(x_{i_1}(1-t_1)+\sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\dots t_j + x_{i_n} t_1\dots t_{n-1}\Big) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i). \tag{3}$$

Indeed, we have

$$\begin{aligned}
\frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) &= \frac{1}{P_n^n}\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n} p_{i_1}\cdots p_{i_n}\Big(f(x_{i_1})(1-t_1) + \\
&\quad + \sum_{j=1}^{n-2} f(x_{i_j})(1-t_{j+1})t_1\cdots t_j + f(x_{i_n})t_1\cdots t_{n-1}\Big) \ge \\
&\ge \frac{1}{P_n^n}\sum_{i_1=1}^{n}\cdots\sum_{i_n=1}^{n} p_{i_1}\cdots p_{i_n} f\Big(x_{i_1}(1-t_1) + \\
&\quad + \sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\cdots t_j + x_{i_n}t_1\cdots t_{n-1}\Big) \ge \\
&\ge \frac{1}{P_n^{n-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_{n-1}=1}^{n} p_{i_1}\cdots p_{i_{n-1}} f\Big(x_{i_1}(1-t_1) + \\
&\quad + \sum_{j=1}^{n-2} x_{i_j}(1-t_{j+1})t_1\cdots t_j + \bar{x}t_1\cdots t_{n-1}\Big) \ge \\
&\ge \frac{1}{P_n^{n-2}}\sum_{i_1=1}^{n}\cdots\sum_{i_{n-2}=1}^{n} p_{i_1}\cdots p_{i_{n-2}} f\Big(x_{i_1}(1-t_1) + \\
&\quad + \sum_{j=1}^{n-3} x_{i_j}(1-t_{j+1})t_1\cdots t_j + \bar{x}t_1\cdots t_{n-2}\Big) \\
&\ \ \vdots \\
&\ge \frac{1}{P_n}\sum_{i_1=1}^{n} p_{i_1} f\big(x_{i_1}(1-t_1)+\bar{x}t_1\big) \ge \\
&\ge f\Big(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\Big).
\end{aligned}$$

REFERENCE

[1] Dragomir S.S.: An improvement of Jensen's inequality, Matematički Bilten, to appear

Josip Pečarić,
Faculty of Technology,
Zagreb, Yugoslavia