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ON AN EXAMPLE OF FINITEDIMENSIONAL ALGEBRAS

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Abstract. An algebra A over a field K can be finitedimensional or infinitedimensional. If the multiplication in A is commutative than the algebra A itself is said to be commutative.

In this paper we formulate two theorems: the first theorem concerns the commutative finitedimensional algebras and the second one, in fact, generalizes one example of Banach algebras whose factor algebra is finitedimensional and is of the type described in the first theorem.

Let us recall some of the basic notions and properties, mainly concerning Banach algebras, in order to make our proofs and formulations more concise.

If A is a *Banach vector space* with respect to a norm that satisfies the multiplicative inequality

$$||xy|| \leq ||x||\cdot||y|| \qquad (x\in A,\ y\in A)$$

then A is called a *Banach algebra*. Here we are mainly concerned with Banach algebras over the complex field $\mathbb{C}$. We presume, as well, that the algebra A contains a multiplicative *unit element*; which does not mean any special limitation ([1], page 8). A vector subspace J of a commutative Banach algebra A is said to be an *ideal* if $xy\in J$ whenever $x\in A$ and $y\in J$. If $J \neq A$, J is a *proper* ideal. *Maximal ideals* are proper ideals which are not contained in any larger proper ideal. If A is commutative Banach algebra, then every maximal ideal of A is closed. Further, each maximal ideal is the kernel of some non-zero complex homomorphism of A and conversely, the kernel of each complex homomorphism is a maximal ideal of A. The set Δ of all maximal ideals of an algebra A, equipped with its Gelfand topology is a compact Hausdorff space which is usually called the *maximal ideal space* of A. The *radical* of A, denoted by rad A, is the intersection of all maximal ideals of A. If rad $A = \{0\}$, A is called *semisimple*.

Suppose J is a proper closed ideal in a commutative Banach algebra A and $\pi: A \to A/J$ is the quotient map. Then A/J is a commutative Banach algebra and π is a homomorphism.

In the theorem which follows we give sufficient conditions for a finitedimensional algebra to be commutative:

Theorem 1. Let A be a finitedimensional algebra over a field K. If A has base $B = \{e_0, e_1, \ldots, e_n\}$ where e_0 is the multiplicative unit element and $e_k = (e_1)^k$, k=1,2,...,n and $e_1^{n+1} = 0$, then A is commutative.

Proof. Let $a,b \in A$, then:

$$a = \sum_{k=0}^{n} \lambda_k e_k, \quad b = \sum_{j=0}^{n} \mu_j e_j \quad \text{and} \quad ab = \sum_{k=0}^{n} \lambda_k \sum_{j=0}^{n} \mu_j e_k e_j .$$

Since, from the hypothesis of the theorem, $e_k e_j = e_j e_k$, it follows that ab = ba.

As a realization of algebras of the above described type we can quote the Jordan matrixes

$$\begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{n-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \alpha_1 \end{bmatrix}$$

with the base

$$E_0 = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \; E_1 = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}, \ldots, \; E_n = \begin{bmatrix} 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix}.$$

In the following theorem we give a concrete construction of algebras of the type described in Theorem 1. Theorem 2, in fact, generalizes an elementary example (see example 9, [2], page 288) of a Banach algebra $A = C^1[0,1]$ which is an algebra of all continuously differentiable complex functions on the unit interval [0,1] with pointwise multiplication, normed by

$$||f|| = ||f||_\infty + ||f'||_\infty, \text{ where } ||f||_\infty = \sup_{x \in [0,1]} |f(x)| .$$

Theorem 2. Let $A = C^n[0,1]$ be the algebra of all n-fold continously differentiable complex functions on the unit interval [0,1], with pointwise multiplication, normed by

$$||f|| = ||f||_\infty + ||f'||_\infty + \ldots + ||f^{(n)}||_\infty .$$

Then A is a semisimple commutative Banach algebra with maximal ideal space [0,1]. If $J = \{f \in A \mid f(0) = f'(0) = \ldots = f^{(n)}(0) = 0\}$ then J is a closed ideal in A and A/J is a finitedimensional algebra that is not semisimple. In fact, A/J is an algebra as that one of Theorem 1.

Proof. It is obvious that A is commutative Banach algebra. Each maximal ideal of A is of the form $J = \{f \in A \mid f(p) = 0\}$ for $0 \le p \le 1$. Hence, $f \in \mathrm{rad}A$ if f annihilate in every point of [0,1], that means $f(x) = 0$ for all $x \in [0,1]$. So A is semisimple. The proof that the maximal ideal space is the unit interval [0,1] is the same as in the example mentioned before the Theorem 2.

Obviously, the set J is a closed ideal, since if $f \in \bar{J}$, then there exists a sequence $\{f_n(x)\}$ in J such that $||f_n - f|| \to 0$ when $n \to \infty$, which implies that $\lim_{n\to\infty} f_n^{(k)}(0) = f^{(k)}(0) = 0$, $k=0,1,\ldots,n$. It is obvious that J is not a maximal ideal, since $J \subset J_0$.

Now, let us turn our attention to the quotient algebra A/J. If $f \in A$, then the function

$$g(x) = f(x) - f(0)e_0(x) - \frac{f'(0)}{1!}e_1(x) - \ldots - \frac{f^{(n)}(0)}{n!}e_n(x) \qquad (1)$$

where $e_0(x) = 1$, $e_1(x) = x$, ..., $e_n(x) = x^n$, belongs to the ideal J. It follows, from (1) that

$$f(x) = f(0)e_0(x) + \frac{f'(0)}{1!}e_1(x) + \ldots + \frac{f^{(n)}(0)}{n!}e_n(x) + g(x)$$

If we apply the quotient map π on f, we have:

$$\Pi(f) = f(0)\Pi(e_0) + \frac{f'(0)}{1!}\Pi(e_1) + \ldots + \frac{f^{(n)}(0)}{n!}\Pi(e_n). \qquad (2)$$

Hence, we have that every vector $\pi(f)$ from the quotient algebra A/J can be expressed as a linear combination of the vectors

$$\pi(e_0), \pi(e_1), \ldots, \pi(e_n) \qquad (3)$$

which are independent, since on the contrary, there will existe complex numbers $\alpha_o,\alpha_1,\ldots,\alpha_n$ such that

$$\alpha_o\pi(e_o) + \alpha_1\pi(e_1) + \ldots + \alpha_n\pi(e_n) = 0$$

or,

$$\pi(\alpha_o e_o + \alpha_1 e_1 + \ldots + \alpha_n e_n) = 0,$$

which implies that $\alpha_o 1 + \alpha_1 x + \ldots + \alpha_n x^n \in J$, but this is impossible if at least one of the numbers $\alpha_o,\alpha_1,\ldots,\alpha_n$ is not equal to zero. This proves that the vectors (3) present a base for the quotient algebra A/J, and, taking into account that π is a homomorphism, we have that

$$\pi(e_k) = \pi(e_1^k) = [\pi(e_1)]^k, \quad k=1,2,\ldots,n$$

and, since $x^{n+1}\in J$, $\pi(e_1^{n+1}) = \pi(x^{n+1}) = 0$. This proves that finitedimensional algebra A/J is of type described in Theorem 1.

Now, let us turn at the vector space M generated by the vectors, $\pi(e_1),\ldots,\pi(e_n)$. It is easy to verify that M is an ideal with codimension 1, i.e. is a maximal ideal. Let us presume that M_1 is a maximal ideal in A. Since the codimension of M_1 is one, it posseses a base of n vectors, for example $b_1,b_2,\ldots,b_n$. Each of the vectors b_j has a unique presentation in the base (3):

$$b_j = \sum_{k=o}^{n} \beta_k^{(j)} \Pi(e_k), \quad j=1,2,\ldots,n \tag{4}$$

Let us presume that in the vector $b_1 = \beta_o^{(1)}\pi(e_o) +\ldots+ \beta_n^{(1)}\pi(e_n)$ $\beta_o^{(1)} \neq 0$. If we multiply b_1 by $\pi(e_n)$ we have

$$\pi(e_n)b_1 = \beta_o^{(1)}\pi(e_n)$$

which implies that $\pi(e_n)\in M_1$, since M_1 is an ideal. Hence, if we multiply

$$b_1 - \beta_n^{(1)}\pi(e_n) = \beta_o^{(1)}\pi(e_o) +\ldots+ \beta_{n-1}^{(1)}\pi(e_{n-1})$$

by $\pi(e_{n-1})$, we get

$$b_1\pi(e_{n-1}) = \beta_o^{(1)}\pi(e_{n-1}) + \beta_1^{(1)}\pi(e_n)$$

which implies that $\pi(e_{n-1})\in M_1$. Continuing in this way, at the end we come to the conclusion that $\pi(e_k)\in M_1$, k=n,n-1,...,1, which means that M_1 is not a proper ideal. If, on the other hand,

$\beta_o^{(j)} = 0$ for $j=1,2,\dots,n$ then $M_1 \in M$. Hence, we have that the only maximal ideal in the algebra A/J is the ideal M, i.e. $\mathrm{rad}(A/J)=M$, which means that A/J is not a semisimple algebra even though the algebra A is semisimple.

Remark. From the proof given above immediately follows that every proper ideal of quotient algebra A/J is nilpotent. Hence, we also get that A/J is not semisimple ([1], page 35).

R E F E R E N C E S

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ЗА ЕДЕН ПРИМЕР ОД КОНЕЧНОДИМЕНЗИОНАЛНИТЕ АЛГЕБРИ

Н. Речкоски и А. Бучковска

Р е з и м е

Во овој текст дадени се две теореми. Теоремата 1 е од конечнодимензионалните алгебри и во неа се дадени услови за да важи комутативниот закон при множењето. Во Теоремата 2 е опишан еден начин на конструкција на алгебри од типот на алгебри како во Теорема 1. Инаку, иако Теоремата 2 е зависна од Банаховите алгебри сепак сметаме дека и во неа не помалку е од интерес алгебарскиот аспект во однос на тополошкиот.

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