MOIRÉ FRINGES OF TWO SYSTEMS OF CONCENTRIC HOMOTHETIC SECOND ORDER CURVES WHOSE SEMIAXES REFER TO EACH OTHER AS THE SQUARE ROOTS OF THE INTEGERS

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1). INTRODUCTION

In an earlier article[1] we have discussed the moiré fringes obtained by overlapping the two systems of concentric and homothetic ellipses, whose semiaxes refer to each other as square roots of the integers. This article was a generalisation of our previous research about the moiré fringes of two systems of concentric circles with the radiai related to each other as square roots of the integers, i.e. the fringes obtained from two Sorret's zone plates[2]. In[1] we restricted our generalisation to ellipses only. Here we'll extend the discussion to two systems of hyperbolas or to one system of ellipses and one system of hyperbolas.

2). THE EQUATIONS OF THE SYSTEMS OF CURVES AND THE SYSTEM OF MOIRÉ FRINGES

Let α and b be the semiaxes of the smallest curve in one of the systems. Than the semiaxes of the n-th curve in that system are $a\sqrt{n}$ and $b\sqrt{n}$. The axial equation of such a system of homothetic ellipses is given by

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = n \qquad n = 1, 2, 3, \ldots \tag{1}$$

while for the system of the hyperbolas we have the equation

$$\pm \frac{\xi^2}{a^2} \mp \frac{\eta^2}{b^2} = n \tag{2}$$

Here we put double symbols in order to have hyperbolas in each of the four angles bounded by their assipthotes, since we need a system of curves covering the whole plane.

In order to deal simultaneously with both types of curves we introduce the following marks

$$g = \begin{cases} 1 & \text{for ellipses} \\ \pm 1 & \text{for hyperbolas} \end{cases} \tag{3}$$

and

$$e = \begin{cases} 1 & \text{for ellipses} \\ -1 & \text{for hyperbolas} \end{cases} \tag{4}$$

Than both systems (1) and (2) are given by the equation

$$g\left(\frac{\xi^2}{a^2} + e\frac{\eta^2}{b^2}\right) = n \tag{5}$$

Let's shift the origin of the coordinate system to the right for **s** and rotate the coordinate axes for an angle α. Such translation and rotation leed to the followong transformation formulas

$$\begin{aligned} \xi &= (x - s)\cos\alpha - y\sin\alpha \\ \eta &= (x - s)\sin\alpha + y\cos\alpha \end{aligned} \tag{6}$$

The equation of the rotated system of curves is given by

$$Kx^2 + 2\,Lxy + My^2 - 2\,Ksx - 2Lsy + Ks^2 = n \tag{7}$$

where K, L and M are the substitutes as follows

$$\begin{aligned} K &= g\,(a^{-2}\cos^2\alpha + eb^{-2}\sin^2\alpha) \\ L &= g\,(-a^{-2} + eb^{-2})\sin\alpha\ \text{sco}\ \alpha \\ M &= g\,(a^{-2}\sin^2\alpha + eb^{-2}\cos^2\alpha) \end{aligned} \tag{8}$$

If the second system of curves is rotated and translated to the left, it' ll have the opposite sign in front of **s** in it's equation.

Indexing by 1 and 2 the characteristic values, we get the systems' equations

$$\begin{aligned} K_1 x^2 + 2\,L_1 xy + M_1 y^2 - 2\,K_1 sx - 2\,L_1 sy + K_1 s^2 &= n_1 \\ K_2 x^2 + 2\,L_2 xy + M_2 y^2 + 2\,K_2 sx + 2\,L_2 sy + K_2 s^2 &= n_2 \end{aligned} \tag{9}$$

The overlapping of the systems gives their moiré fringes connected by the indicial equation

$$n_1 + \varepsilon n_2 = p \qquad\qquad p = 1, 2, 3, \ldots \tag{10}$$

where

$$p = \begin{cases} 1, 2, 3 \dots \\ \pm 1, \pm 2, \pm 3 \dots \end{cases} \qquad \varepsilon = \begin{cases} 1 & \text{for additive fringes} \\ -1 & \text{for subtractive fringes} \end{cases}$$

Using (9) we find the equation of the moiré fringes to be

$$Ax^2 + 2Bxy + Cy^2 + 2\,Dx + 2\,Ey + As^2 = p \tag{11}$$

with

$$\begin{aligned} A &= K_1 + \varepsilon K_2 & D &= -s\,(K_1 - \varepsilon\, K_2) \\ B &= L_1 + \varepsilon L_2 & E &= -s\,(L_1 - \varepsilon\, L_2) \\ C &= M_1 + \varepsilon\, M_2 \end{aligned} \tag{12}$$

The equation (11) shows that the moiré fringes are a system of second order curves.

3) THE FORM OF THE MOIRE FRINGES SYSTEM

In order to find out what kind of curves are the moiré fringes (11) we need to know the sign of the determinante

$$\Delta = AC - B^2 \tag{13}$$

since, as it is well known, the curves are

$$\begin{matrix} \text{ellipses} \\ \text{parabolas} \\ \text{hyperbolas} \end{matrix} \qquad \text{if } \Delta \begin{cases} > 0 \\ = 0 \\ < 0 \end{cases} \tag{14}$$

Replacing the values (12) in (13) we have

$$\Delta = K_1\,M_1 - L_1^2 + \varepsilon^2\,(K_2\,M_2 - L_2^2) + \varepsilon\,(K_2\,M_1 + K_1 M_2 - 2L_1 L_2) \tag{15}$$

and using relations (8) we have

$$\begin{aligned} \Delta = \frac{1}{a_1^2\,a_2^2\,b_1^2\,b_2^2} &[(a_2^2 + \varepsilon g_1\,g_2\,a_1^2)\,(e_1\,b_2^2 + \varepsilon g_1\,g_2\,e_2\,b_1^2) + \\ &+ \varepsilon\,g_1\,g_2\,(b_1^2 - e_1\,a_1^2)\,(b_2^2 - e_2\,a_2^2)\,\sin^2(\alpha_1 - a_2)] \end{aligned} \tag{16}$$

The sign of this determinante, besides on semiaxes (a_1, a_2, b_1, b_2) and the types of curves (e_1, e_2, g_1, g_2), depends on the angle between the principal axes of the defined systems of curves.

Moiré fringes appear as

$$\begin{array}{l} \text{ellipses} \\ \text{parabolas} \\ \text{hyperbolas} \end{array} \quad \text{if } \sin(\alpha_1 - \alpha_2) \begin{array}{c} < \\ = \\ > \end{array} \sqrt{\frac{(a_2^2 + \varepsilon g_2 a_1^2)(e_1 b_2^2 + \varepsilon g_1 g_2 e_2 b_1^2)}{-\varepsilon g_1 g_2 (b_1^2 - e_1 a_1^2)(b_2^2 - e_2 a_2^2)}} \tag{17}$$

When we owerlap the systems of the same kind, $g_1 g_2 = 1$ and $e_1 = e_2 = e$. If they are of different kind, by choosing the system of the ellipse to have its centre with positive apscise, we put $g_1 g_2 = \pm 1 = g_2$ and $e_2 = -e_1$. In this case the subtractive fringes, where $g_2 = 1$, and the additive, where $g_2 = -1$, give the same value of the square root in (17),as well as the additive fringes where $g_2 = 1$ and the subtractive fringes where $g_2 = -1$. Having in mind the relations (8) and (12), it could be easily verified that the coeficients A, B, C, D, E for the subtractive moire finges in the region where, $g_2 = 1$ are equal with the coefficie nts of the additive moirè fringes in the region where $g_2 = -1$. Therefore this two different kind of moirè fringes are given with one analitical expression, representing one system of curves which are visible in two different regions. Also the subtractive moirè fringes where $g_2 = 1$ and the additive fringes where $g_2 = -1$ belong to the same system of curves. As it will be seen later, the subtractive and additive fringes appearing in one region are devided, by the so called commutation moire boundary.

A very important case in dealing with the moirè fringes, is the case when they appear as a system of paralell lines, since it is the most easiest way of their observation and registration. So, our problem is to answer the question when the equation (11) will represent a system of parallel lines. It generaly happens when the coefficints A, B, and C are equal to zero, provided $D \neq 0$ and $E \neq 0$. Therefore

$$g_1 (a_1^{-2} \cos^2 \alpha_1 + e_1 b_1^{-2} \sin^2 \alpha_1) = -\varepsilon g_2 (a_2^{-2} \cos^2 \alpha_2 + e_2 b_2^{-2} \sin^2 \alpha_2)$$

$$g_1 (-a_1^{-2} + e_2 b_1^{-2}) \sin \alpha_1 \operatorname{soc} \alpha_1 = -\varepsilon g_2 (-a_2^{-2} + e_2 b_2^{-2}) \sin \alpha_2 \cos \alpha_2$$

$$g_1 (a_1^{-2} \sin^2 \alpha_1 + e_1 b_1^{-2} \cos^2 \alpha_1) = -e g_2 (a_2^{-2} \sin^2 \alpha_2 + e_2 b_2^{-2} \cos^2 \alpha_2)$$

By adding this three relations we get the condition

$$g_1 \left[\left(\frac{e_1}{b_1^2} + \frac{1}{a_1^2} \right) + \frac{1}{2} \sin 2\alpha_1 \cdot \left(\frac{e_1}{b_1^2} - \frac{1}{a_1^2} \right) \right] = \\ = -\varepsilon g_2 \left[\left(\frac{e_2}{b_1^2} + \frac{1}{a_2^2} \right) + \frac{1}{2} \sin 2\alpha_2 \cdot \left(\frac{e_2}{b_2^2} - \frac{1}{a_2^2} \right) \right] \tag{18}$$

In the case of overlapping two parallel systems $\alpha_1 = \alpha_2$, this condition is valid only for the subtractive fringes, requiring the systems to be of the same kind and to have the same axes in addition.

Supposing the condition (18) is valid, the equation of the moiré fringes is

$$Dx + Ey - \tfrac{1}{2} p = 0 \tag{19}$$

representing a system of equidistant lines with a separation

$$d = \frac{1}{2\sqrt{D^2 + E^2}} \tag{20}$$

Another case of obtaining the moirè fringes as a system of parallel lines in when

$$D = E = 0 \quad \text{and } B^2 = AC \tag{21}$$

It happens when the overlapping systems have the origin as their centers, i. e. when $s = 0$. The equation of the moirè fringes than is

$$y = -\sqrt{A/C}\; x \pm \sqrt{p/C} \tag{22}$$

a system of parallel lines whose distances from the origin reffer to each other as square roots of the integers.

4) POSITION OF THE MOIRÈ FRINGES' SYSTEM

The position of the moirè fringes' system is determined by the direction of their principal axes and by the position of their center, if the fringes consist of curves having a common center.

The angle ϑ closed between the principal axis of the moirè system of curves, and the apscise axis of the coordinate system, is defined by

$$\operatorname{tg} 2\vartheta = \frac{2B}{C - A} =$$

$$= \frac{a_2^2 b_2^2 (e_1 a_1^2 - b_1^2) \sin 2\alpha_1 + \varepsilon g_1 g_2 a_1^2 b_1^2 (e_2 a_2^2 - b_2^2) \sin 2\alpha_2}{a_2^2 b_2^2 (e_1 a_1^2 - b_1^2) \cos 2\alpha_1 + \varepsilon g_1 g_2 a_1^2 b_1^2 (e_2 a_2^2 - b_2^2) \cos 2\alpha_2} \tag{23}$$

The coordinates of the centre of curves (11) are defined by the well known formulas

$$x_0 = \frac{1}{\Delta} (BE - CD)$$

$$y_0 = \frac{1}{\Delta} (BD - AE) \tag{14}$$

which do not include the order number **p** of the moirè fringes. Therefore, all of the fringes have one common center. According to (12) and (8)

$$x_0 = \frac{s}{\Delta}\{e_1 a_1^{-2} b_1^{-2} - e_2 a_2^{-2} b_2^{-2} + \varepsilon g_1 g_2 [(- a_1^{-2} a_2^{-2} +$$

$$+ e_1 e_2 b_1^{-2} b_2^{-2}) \sin(\alpha_1 + \alpha_2) \sin(\alpha_1 - \alpha_2) + (e_2 a_1^{-2} b_2^{-2} -$$

$$- e_1 a_2^{-2} b_1^{-2}) \cos(\alpha_1 + \alpha_2) \cos(\alpha_1 - \alpha_2)]\}$$

(25)

$$y_0 = \frac{s}{\Delta} \varepsilon g_1 g_2 [(a_2^{-2} \cos^2 \alpha_2 + e_2 b_2^{-2} \sin^2 \alpha_2)(-a_1^{-2} +$$

$$+ e_1 b_1^{-2}) \sin 2\alpha_1 - (a_1^{-2} \cos^2 \alpha_1 +$$

$$+ e_1 b_1^{-2} \sin^2 \alpha_1)(- a_2^{-2} + e_2 b_2^{-2}) \sin 2\alpha_2].$$

It is seen that the ratio y_0/x_0 does not depend on s. It means that when the centers of our two systems are drawned near or mooved away, the moire fringes' center gets displaced allong a straight line passing througt the origin.

5). THE COMMUTATION MOIRÈ BOUNDARY AND EFFICIENCY OF THE FRINGES

According to [3] the commutation moirè bondary deviding the regions of effectivness of the additive and subtractive moirè fringes, is defined by the condition

$$\text{grad}\ \psi_1 . \ \text{grad}\ \psi_2 = 0 \tag{26}$$

turing into

$$\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial x} + \frac{\partial\psi_1}{\partial y}\frac{\partial\psi_2}{\partial y} = 0 \tag{27}$$

for the two dimensional case of Decart's coordinates. ψ_1 and ψ_2 are the analitic expressions of the systems of curves (9) defined as

$$\psi_1 \equiv K_1 x^2 + 2L_1 xy + M_1 y^2 - 2K_1 sx - 2L_1 sy + K_1 s^2 - n_1 = 0$$

$$\psi_2 \equiv K_2 x^2 + 2L_2 xy + M_2 y + 2K_2 sx + 2L_2 sy + K_2 s^2 - n_2 = 0$$

Therefore the equation of the moirè boundary is

$$(K_1 K_2 + L_1 L_2) x^2 + [L_1 (K_2 + M_2) + L_2 (K_1 + M_1) xy + (L_1 L_2 +$$
$$+ M_1 M_2) y^2 + s [L_1 (K_2 - M_2) - L_2 (K_1 - M_1)] y - s^2 (K_1 K_2 + L_1 L_2) = 0 \tag{28}$$

The commutation moirè boundery is a second order curve with it's center laying on the y-axis. It passes the points ($\pm$ s, 0) i. e. throught the centers of the two systems of curves. What kind of second order curve is the moirè boundary, we find by discussing the sign of it's determinante

$$\Delta_b = (K_1 K_2 + L_1 L_2)(L_1 L_2 + M_1 M_2) - \tfrac{1}{4}[L_1(K_2 + M_2) + L_2(K_1 + M_1)]^2$$

which according to the expressions (8) yelds

$$\begin{aligned}\Delta = \{ & [(a_1^{-2} a_2^{-2} \cos\alpha_1 \cos\alpha_2 + e_1 e_2 b_1^{-2} b_2^{-2} \sin\alpha_1 \sin\alpha_2)(\cos(\alpha_1 - \alpha_2) + \\ & + (e_1 b_1^{-2} a_2^{-2} \sin\alpha_1 \cos\alpha_2 - e_2 b_2^{-2} a_1^{-2} \cos\alpha_1 \sin\alpha_2) \sin(\alpha_1 - \alpha_2)]^2 + \\ & + [(a_1^{-2} a_2^{-2} \cos\alpha_1 \cos\alpha_2 + e_1 e_2 b_1^{-2} b_2^{-2} \sin\alpha_1 \sin\alpha_2) \cos(\alpha_1 - \alpha_2) + \\ & + (e_1 a_2^{-2} b_1^{-2} \sin\alpha_1 \cos\alpha_2 - e_2 b_2^{-2} a_1^{-2} \cos\alpha_1 \sin\alpha_2) \sin(\alpha_1 - \alpha_2)] \\ & + [(a_1^{-2} + e_1 b_1^{-2})(a_2^{-2} + e_2 b_2^{-2}) - (a_2^{-2} + e_2 b_2^{-2})(a_1^{-2} \cos^2\alpha_1 + \\ & + e_1 b_1^{-2} \sin^2\alpha_1) - (a_1^{-2} + e_1 b_1^{-2})(a_2^{-2} \cos^2\alpha_2 + e_2 b_2^{-2} \sin^2\alpha_2)] - \\ & - \frac{1}{16} [(e_1 b_1^{-2} - a_1^{-2})(a_2^{-2} + e_2 b_2^{-2} \sin 2\alpha_1 + (e_2 b_2^{-2} - \\ & - a_2^{-2})(a_1^{-2} + e_1 b_1^{-2}) \sin 2\alpha_2]^2 \end{aligned} \tag{29}$$

Since g_1 and g_2 do not enter in the expression (29), it means that if there exists a moirè boundary, it is defined in the whole plane of observation, which is important when one of the two systems or both of them consists of hyperbolas.

As it can be seen, Δ_b in (29) can be bigger, equal or smaller than zero. Therefore the commutation moirè boundary appears as an ellipse, parabola or hyperbola. Inside this boundary visible are the additive fringes, while outside it the subtractive fringes.

6. MOIRÈ FRINGES OF THE SYSTEMS WITH PARALLEL AXES

It is the case when $\alpha_1 = \alpha_2 = \alpha$. The moirè fringes' equation remaines (11), and it's form is determined by the sign of (16), which for this special case is

$$\begin{aligned} \Delta_a &= \frac{1}{a_1^2 a_2^2 b_1^2 b_2^2} (a_2^2 + g_1 g_2 a_1^2)(e_1 b_2^2 + g_1 g_2 e_2 b^2{}_2) \qquad \varepsilon = 1 \\ \Delta_s &= \frac{1}{a_1^2 a_2^2 b_1^2 b_1^2} (a_2^2 - g_1 g_2 a_1^2)(e_1 b_2^2 - g_1 g_2 e_2 b_1^2), \quad \varepsilon = -1 \end{aligned} \tag{30}$$

for the additive and subtractive case respectivly.

Let us first discuss the case of overlapping the systems of the same kind of curves, i.e. $g_1 \cdot g_2 = 1$ and $e_1 = e_2 = e$.

From (30) it follows that the additive moirè fringes are

$$\begin{array}{ll} \text{ellipses} & \\ \text{hyperbolas} & \end{array} \quad \text{if} \quad e \begin{cases} = 1 \\ = -1 \end{cases} \tag{31}$$

The form of the subtractive fringes is

$$\begin{array}{llll} \text{ellipse} & & a_1 < a_2;\ b_1 < b_2 & \\ \text{hyperbola} & \text{if} & \begin{cases} a_1 < a_2;\ b_1 > b_2 \\ a_1 > a_2;\ b_1 < b_2 \end{cases} & e = 1 \\ \text{ellipse} & & \begin{cases} a_1 < a_2;\ b_1 > b_2 \\ a_1 > a_2;\ b_1 < b_2 \end{cases} & \\ \text{hyperbolas} & \text{if} & a_1 < a_2;\ b_1 < b_2 & e = -1 \end{array} \tag{32}$$

No matter whether we overlap two systems of ellipse sor two systems of hyperbolas, the subtractive moirè firnges will be parabolas if $a_1 = a_2$ or $b_1 = b_2$.

The second possibility of overlapping the systems of different kind requires $g_1 \cdot g_2 = \pm 1 = g_2$ and $e_2 = -e_1 (e_1 = 1)$.

From (30) it follows that the additive moirè finges are

$$\begin{array}{llll} & \text{ellipses} & & b_2 > b_1 & \\ & \text{hyperbolas} & \text{if} & b_2 < b_1 & \text{where } g_2 = 1 \\ & \text{parabolas} & & b_2 = b_1 & \\ \text{and} & & & & \\ & \text{ellipses} & & a_2 > a_1 & \\ & \text{hyperbolas} & \text{if} & a_2 < a_1 & \text{where } g_2 = -1 \\ & \text{parabolas} & & a_2 = a_1 & \end{array} \tag{33}$$

The subtractive fringes appear as

$$\begin{array}{llll} & \text{ellipses} & & a_2 > a_1 & \\ & \text{hyperbolas} & \text{if} & a_2 < a_1 & \text{where } g_2 = 1 \\ & \text{parabolas} & & a_2 = a & \\ \text{and} & & & & \\ & \text{ellipses} & & b_2 > b_1 & \\ & \text{hyperbolas} & \text{if} & b_2 < b_1 & \text{where } g_2 = -1 \\ & \text{parabolas} & & b_2 = b_1 & \end{array} \tag{34}$$

The orientation of the principal axis of the moirè fringes defined by the relation (23) remains unchanged, no matter what kind of systems we overlap, since for $\alpha_1 = \alpha_2$, $\theta = \alpha$. It means the axes of the moirè fringes will also be parallel with the axes of the overlapping systems.

Thè position of the center of the moirè fringes depends on the kind of curves, and it's position is determined by

$$x_0 = \frac{s}{\Delta}\,[e_1 a_1^{-2} b_1^{-2} - e_2 a_2^{-2} b_2^{-2} + \varepsilon g_1 g_2 (e_2 a_1^{-2} b_2^{-2} - e_1 a_2^{-2} b_1^{-2}) \cos 2\alpha]$$
$$y_0 = \frac{s}{\Delta}\,\varepsilon g_1 g_2 (e_1 a_2^{-2} b_1^{-2} - e_2 a_1^{-2} b_2^{-2}) \sin 2\alpha. \quad (35)$$

xince $\Delta_a \neq \Delta_b$ the center of the additive moirè fringes doesn't coincide with that of the subtractive fringes.

The equation of the moirè boundary defined by the relaton (28) now has it's determinant given by

$$\Delta_b = e_1 e_2 a_1^{-2} a_2^{-2} b_1^{-2} b_2^{-2} \quad (36)$$

If bouth systems are of the same kind, $e_1 . e_2 = 1$, and from (36) it follows that $\Delta_b > 0$. Therefore their moirè boundary is an ellipse. If the two overlapping systems are of different kind, $e_1 . e_2 = -1$, $\Delta_b < 0$ and their moirè boundary is a hyperbola.

Inside the region bounded by the commutation moirè boundary are visible the additive fringes (31) or (33), while otside it we have the subtractive fringes (32) or (34).

7. MOIRÈ FRINGES OF THE SYSTEMS HAVING EQUAL AXES

The overlapping systems are such that $a_1 = a_1$ and $b_1 = b_2$. The obtained moirè fringes are given with an equation of the from (11) with determinants

$$\Delta_a = \frac{1}{a^4 b^4}\,[a^2 b^2 (1 + g_1 g_2)(e_1 + g_1 g_2 e_2) + $$
$$+ g_1 g_2 (b^2 - e_1 a^2)(b^2 - e_2 a^2) \sin^2 (\alpha_1 - \alpha_2) \quad (37)$$
$$\Delta_b = \frac{1}{a^4 b^4}\,[a^2 b^2 (1 - g_1 g_2)(e_1 - g_1 g_2 e_2) -$$
$$- g_1 g_2 (b^2 - e_1 a^2)(b^2 - e_2 a^2) \sin^2 (\alpha_1 - \alpha_2)$$

In the case of overlapping the systems of the same kind, ($g_1 = g_2$; $g_1 g_2 = 1$; $e_1 = e_2$) the additive fringes are

ellipses			$<$
parabolas	if	$\sin(\alpha_1 - \alpha_2)$	$= ab\sqrt{\dfrac{2e}{ea^2 - b^2}}$ (38)
hyperbolas			$>$

The subtractive fringes are hyperbolas, no matter what kind of curves the two overlapping systems cosist of, since

$$\Delta_0 = -\frac{1}{a^4 b^4}(b^2 - ea^2)^2 \sin^2(\alpha_1 - \alpha_2) < 0.$$

If in addition the axes of the two systems are parallel, the aditive moirè fringes are ellipses ($e = 1$) and hyperbolas ($e = -1$) since $\Delta = 2\, ea^{-2} b^{-2}$.

For the substractive finges we get $\Delta_1 = 0$, but they are not parabolas since the condition (18) is valid. The equation of the subtractive fringes is

$$(b^2 \cos^2\alpha + e\, n^2 \sin^2\alpha)\, x + (ea^2 - b^2) \sin\alpha \cos\alpha \cdot y + pa^2 b^2/4sg = 0 \qquad (39)$$

representing a system of equidistant parallel lines. The distance between two of the lines is

$$d = \frac{a^2 b^2}{4\, s \sqrt{b^4 \cos^2\alpha + a^4 \sin^2\alpha^4}} \qquad (40)$$

The commutative moirè boundary has (23) as it's analitical expression. It is a second order curve with a determinante

$$\begin{aligned}\Delta_b = \{ &[(a^{-4} \cos\alpha_1 \cos\alpha_2 + b^{-4} \sin\alpha_1 \sin\alpha_2) \cos(\alpha_1 - \alpha_2) + ea^{-2} b^{-2} \sin^2(\alpha_1 - \\ &- \alpha_2)]^2 + [(a^{-4} \cos\alpha_1 \cos\alpha_2 + b^{-4} \sin\alpha_1 \sin\alpha_2) \cos(\alpha_1 - \alpha_2) + \\ &+ ea^{-2} b^{-2} \sin^2(\alpha_1 - \alpha_2)]\,(b^{-4} - a^{-4})(\sin^2\alpha_1 - \cos^2\alpha_2) - \\ &- \frac{1}{16}(b^{-4} - a^{-4})^2 (\sin 2\alpha_1 - \sin 2\alpha_2)^2\end{aligned} \qquad (41)$$

The value of this determinante could be positive, negative or equal to zero, and therefore the moirè boundary of the overlapping systems of the same kind having equal axes could be an ellipse, hyperbola or parabola. In the special case when $\alpha_1 = \alpha_2$, the expression (41) is simplified to

$$\Delta_b = a^{-4} b^{-4} > 0 \qquad (42)$$

(which is also expected from (36)), indicating that moirè boundary is an ellipse.

For the other possibility of overlapping the systems of diferent kind, we take $g_1 \cdot g_2 = \pm 1$; g_2; $e_1 = 1$; $e_2 = -1$. Therefore instead of (37) we have

$$\Delta = -\varepsilon g^2 a^{-4} b^{-4} (a^4 - b^4) \sin^2(\alpha_1 - \alpha_2) \qquad (43)$$

Considering $b < a$, the additive fringes appear as

$$\begin{matrix}\text{hyperbolas} \\ \text{ellipses}\end{matrix} \quad \text{where} \quad g = \begin{cases} 1 \\ -1 \end{cases} \qquad (44)$$

while for the subtractive fringes we have the reversed situation. They are

$$\begin{matrix} \text{hyperbolas} \\ \text{ellipses} \end{matrix} \quad \text{where} \quad g = \begin{cases} 1 \\ -1 \end{cases} \tag{45}$$

The moirè fringes appear as parabolas only in the case of parallel overlapping, when $\alpha_1 = \alpha_2$, since for both subtractive and additive fringes A, B and $C \neq 0$, but the determinante (43) is zero. Of course the subtractive fringes where $g_2 = 1$ and the additive fringes where $g_2 = -1$ belong to one system of parabolas, while the additive fringes where $g_2 = 1$ and the subtractive fringes where $g_2 = -1$ belong to another sistem of parabolas.

So far we'll live the disscusion obout the zero value of (43) when, $a = b$, since the moirè fringes obtained by overlapping the systems of circles and equilateral hyperbolas with systems of other homothetic second order curves will be treated in an other article.

There exists a commutation moirè boundary given with an equation of the form (28), having the determinante given by

$$\Delta = [a^{-4} \cos\alpha_1 \cos\alpha_2 - b^{-4} \sin\alpha_1 \sin\alpha_2) \cos(\alpha_1 - \alpha_2) + a^{-2} b^{-2} \sin(\alpha_1 + \alpha_2)$$
$$\sin(\alpha_1 - \alpha_2)]^2 + [(a^{-4} \cos\alpha_1 \cos\alpha_2 - b^{-4} \sin\alpha_1 \sin\alpha_2) \cos(\alpha_1 - \alpha_2) +$$
$$+ a^{-2} b^{-2} \sin(\alpha_1 + \alpha_2) \sin(\alpha_1 - \alpha_2)] [(a^4 - b^4) - (a^{-2} - b^{-2}) (a^{-2} \cos^2\alpha_1 +$$
$$+ b^{-2} \sin^2\alpha_1) - (a^{-2} + b^{-2}) (a^{-2} \cos^2\alpha_2 - b^{-2} \sin^2\alpha_2)] -$$
$$- \frac{1}{16} [(b^{-2} - a^{2-})^2 \sin 2\alpha_1 + (b^{-2} + a^{-2})^2 \sin 2\alpha_2]^2\} \tag{46}$$

It indicates that the moirè boundary could be an ellipse, parabola or a hyperbola. If in addition $\alpha_1 = \alpha_2$.

$$\Delta_b = -a^{-4} b^{-4} < 0 \tag{47}$$

and the commutation boundary appears as an hyperbola.

Inside this moirè boundary are visible the additive fringes, while ottside it, the subtractive fringes.

REFERENCES

1. J. Moser, S. Bahčevandžiev, M. Jonoska, Lj. Janićijević: „Moire figure specijalnih eliptičkih" mrežica" Годишен Зборник на Природно-математичкиот факултет на универзитетот во Скопје, Сек. А, книга 17—18. (1966—1967) Скопје.
2. J. Мозер, С. Бахчеванџиев, М. Јоноска и Љ. Јаниќијевиќ „Муаре фигури на зонските мрежички" Годишен Зборник на Природно-математичкиот факултет на Универзитетот во Скопје, Сек. А, Книга 15 (1964) Скопје.
3. J. Мозер и Љ. Јаниќијевиќ: „За ефективноста на муаре фигурите и нивната комутативна граница" Годишен Зборник на Природно-математичкиот факултет на Универзитетот во Скопје, Книга 23 (1973) Скопје.

МУАРЕ ФИГУРИ НА ДВА СИСТЕМИ КОНЦЕНТРИЧНИ ХОМОТЕТИЧНИ КРИВИ ОД ВТОР РЕД ЧИИ ПОЛУОСКИ СЕ ОДНЕСУВААТ КАКО КВАДРАТНИ КОРЕНИ ОД ЦЕЛИТЕ БРОЕВИ

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Дискутирани се муаре фигурите добиени со преклопување на два системи од концентрични хомотетични хиперболи, како и со преклопување на систем од хомотетични концентрични хиперболи со систем од хомотетични концентрични елипси.

Релацијата (5) заедно со ознаките (3) и (4) претставува општа равенка на n-тата крива од гореспоменатите системи. Таа преминува во (7) кога центарот на кривата е поместен за s од координатниот почеток и оските на системот се ротирани за агол α во однос на координатните оски. Ако е координатниот почеток на средината меѓу центрите на двата системи од криви, тогаш тие се зададени со равенките (9). Равенката на нивните муаре фигури е (11) и претставува систем од криви од втор ред чија природа, според условот (17), зависи од типовите на кривите кои се преклопуваат, од големината на нивните полуоски, како и од аголот зафатен меѓу нивните главни полуоски. Посебно е дискутиран случајот на премин на муаре фигурите во систем од прави. Тоа настанува кога се исполнети условите (18) или (21).

Во делот под точка 4) дадена е кратка дискусија за положбата на системот од муаре фигури. Дефинирани се координатите на нивниот центар (25) како и аголот што главната оска на муаре системот го зафаќа со апсцисната оска (23).

Во делот пак под точка (5) дадена е равенката на комутативната муаре граница (28) која секојпат претставува крива од втор ред. Во внатрешноста на оваа крива видвили се адитивните а надвор од неа суптрактивните муаре фигури.

На крајот во деловите под точките 6) и 7) посебно се дискутирани случаите на добивање на муаре фигури со преклопување на системи со параделни оски и на системи со еднакви оски.