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SOME NEW OSTROWSKI TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR BETA (r, g)-PREINVEX FUNCTIONS VIA CAPUTO k-FRACTIONAL DERIVATIVES

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Abstract. In the present paper, the notion of beta (r, g)-preinvex function is applied to establish some new generalizations of Ostrowski type integral inequalities via Caputo k-fractional derivatives.

1. Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. The set of continuous differentiable functions of order n on the interval $[a, b]$ is denoted by $C^n[a, b]$.

The following result is known in the literature as the Ostrowski inequality [24], which gives an upper bound for the approximation of the integral average $\dfrac{1}{b-a}\displaystyle\int_a^b f(t)dt$ by the value $f(x)$ at point $x \in [a, b]$.

Theorem 1.1. *Let $f : I \longrightarrow \mathbb{R}$ be a mapping differentiable on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then*

$$\Big|f(x) - \frac{1}{b-a}\int_a^b f(t)dt\Big| \leq M(b-a)\Big[\frac{1}{4} + \frac{\big(x - \frac{a+b}{2}\big)^2}{(b-a)^2}\Big], \quad \forall\, x \in [a, b]. \tag{1.1}$$

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For other recent results concerning Ostrowski type inequalities readers are related to, see [5],[15],[17],[20],[24],[27]-[29],[31],[36]-[39]. Ostrowski inequality is playing a very important role in all the fields of mathematics, especially in the theory of approximations. Thus such inequalities were studied extensively by many researches and numerous generalizations, extensions and variants of them for various kind of functions like bounded variation, synchronous, Lipschitzian, monotonic, absolutely, continuous and n-times differentiable mappings etc. appeared in a number of papers, see [26]-[32].

In numerical analysis many quadrature rules have been established to approximate the definite integrals. Ostrowski inequality provides the bounds of many numerical quadrature rules, see [9],[18]. In recent decades Ostrowski inequality is studied in fractional calculus point of view by many mathematicians, see [1]-[3],[7],[8],[10]-[12],[15]-[18],[20],[22]-[25],[27]-[29],[34], [35],[40].
Now, let us evoke some basic definitions.

Definition 1.2. [14] A function $f : [0, +\infty) \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{1.2}$$

for all $x, y \geq 0,\ \lambda \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $[0, +\infty)$ as usual. The s-convex functions in the second sense have been investigated in [14].

Definition 1.3. [4] A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\eta(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not necessarily true, see [4],[38].

Definition 1.4. [33] The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect η, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f\left(x + t\eta(y,x)\right) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\eta(y,x) = y - x$, but the converse is not true.

Definition 1.5. For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow \infty} \frac{n!\, k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{1.3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{1.4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

For $k = 1$, (1.4) gives integral representation of gamma function.

Definition 1.6. For $k \in \mathbb{R}^+$ and $x, y \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt. \tag{1.5}$$

For $k = 1$, (1.5) gives integral representation of beta function.

Definition 1.7. [21] Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and are continuous on $[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$${}^cD^\alpha_{a+} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \; x > a \tag{1.6}$$

and

$${}^cD^\alpha_{b-} f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \; x < b. \tag{1.7}$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $\left({}^cD^\alpha_{a+} f\right)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$\left({}^cD^0_{a+} f\right)(x) = \left({}^cD^0_{b-} f\right)(x) = f(x) \tag{1.8}$$

where $n = 1$ and $\alpha = 0$.

In the following we recall Caputo k-fractional derivatives.

Definition 1.8. [13] Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha]+1$, $f \in C^n[a, b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$${}^cD^{\alpha,k}_{a+} f(x) = \frac{1}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}} dt, \; x > a \tag{1.9}$$

and

$$ {}^{c}D_{b-}^{\alpha,k} f(x) = \frac{(-1)^n}{k\Gamma_k\left(n - \frac{\alpha}{k}\right)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}} dt, \; x < b. \tag{1.10} $$

The aim of this paper is to establish some generalizations of Ostrowski type inequalities using new integral identity for beta (r, g)-preinvex functions via Caputo k-fractional derivatives.

2. Main results

Definition 2.1. [6] A set $K \subseteq \mathbb{R}$ is said to be m-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ holds for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 2.2. In Definition 2.1, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. For example when $m = 1$, then the m-invex set degenerates an invex set on K.

We next give new definition, to be referred as beta (r, g)-preinvex function.

Definition 2.3. Let $K \subseteq \mathbb{R}$ be an open m-invex set with respect to $\eta : K \times K \longrightarrow \mathbb{R}$, $g : [0, 1] \longrightarrow [0, 1]$ be a differentiable function and $\varphi : I \longrightarrow K$ is a continuous function. The function $f : K \longrightarrow (0, +\infty)$ is said to be beta (r, g)-preinvex with respect to η, if

$$ f\left(m\varphi(x) + g(t)\eta(\varphi(y), m\varphi(x))\right) \leq M_r(mf(\varphi(x)), f(\varphi(y)), p, q, g(t)) \tag{2.1} $$

holds for some fixed $m \in (0, 1]$, for any fixed $p, q > -1$, and for all $x, y \in I, t \in [0, 1]$, where

$$ M_r(mf(\varphi(x)), f(\varphi(y)), p, q, g(t)) $$
$$ = \begin{cases} \left[mg^p(t)(1-g(t))^q f^r(\varphi(x)) + g^q(t)(1-g(t))^p f^r(\varphi(y))\right]^{\frac{1}{r}}, & r \neq 0; \\ \\ [mf(\varphi(x))]^{g^p(t)(1-g(t))^q} [f(\varphi(y))]^{g^q(t)(1-g(t))^p}, & r = 0, \end{cases} $$

is the weighted power mean of order r for positive numbers $f(\varphi(x))$ and $f(\varphi(y))$.

Remark 2.4. In Definition 2.3, it is worthwhile to note that the class of beta (r, g)-preinvex function is a generalization of the class of s-convex in the second sense function given in Definition 1.2. For $p = 0, q = s$ and $g(t) = t$, we get the notion of generalized $(r; s, m, \varphi)$-preinvex function [19]. For $r = 1, p = 0, q = s$ and $g(t) = t$, we get the notion of generalized (s, m, φ)-preinvex function [15]. Also, for $r = 1, p = 0, q = s$, $g(t) = t, \forall\, t \in [0, 1]$ and $\varphi(x) = x$, $\forall x \in I$, we get the notion of generalized (s, m)-preinvex function

[6]. This definition is not vague and now it is easy to justify the relation $m\varphi(x)+g(t)\eta(\varphi(y),m\varphi(x))$ and the roles of these parameters $m,p,q,g(t)$ and variables t.

In this section, in order to present some new Ostrowski type integral inequalities for beta (r,g)-preinvex functions via Caputo k-fractional derivatives, we need the following lemma to obtain our results.

Lemma 2.5. *Let $\alpha>0$, $k\geq 1$ and $\alpha\notin\{1,2,3,\dots\}$, $n=[\alpha]+1$. Suppose $K\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\eta: K\times K\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$. Let $\varphi: I\longrightarrow K$ be a continuous function and $g:[0,1]\longrightarrow[0,1]$ a differentiable function. Assume that $f:K\longrightarrow\mathbb{R}$ is a function on K° such that $f\in C^{n+1}[m\varphi(a),m\varphi(a)+\eta(\varphi(b),m\varphi(a))]$, where $\eta(\varphi(b),m\varphi(a))>0$. Then, we have the following equality for Caputo k-fractional derivatives:*

$$\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),m\varphi(a))}{\eta(\varphi(b),m\varphi(a))}\times[g^{n-\frac{\alpha}{k}}(1)f^{(n)}(m\varphi(a)+g(1)\eta(\varphi(x),m\varphi(a)))$$
$$-g^{n-\frac{\alpha}{k}}(0)f^{(n)}(m\varphi(a)+g(0)\eta(\varphi(x),m\varphi(a)))]$$
$$-\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),m\varphi(b))}{\eta(\varphi(b),m\varphi(a))}\times[g^{n-\frac{\alpha}{k}}(1)f^{(n)}(m\varphi(b)+g(1)\eta(\varphi(x),m\varphi(b)))$$
$$-g^{n-\frac{\alpha}{k}}(0)f^{(n)}(m\varphi(b)+g(0)\eta(\varphi(x),m\varphi(b)))]$$
$$-\frac{n-\frac{\alpha}{k}}{\eta(\varphi(b),m\varphi(a))}\times\Big[\int_{m\varphi(a)+g(0)\eta(\varphi(x),m\varphi(a))}^{m\varphi(a)+g(1)\eta(\varphi(x),m\varphi(a))}(t-m\varphi(a))^{n-\frac{\alpha}{k}-1}f^{(n)}(t)dt$$
$$-\int_{m\varphi(b)+g(0)\eta(\varphi(x),m\varphi(b))}^{m\varphi(b)+g(1)\eta(\varphi(x),m\varphi(b))}(t-m\varphi(b))^{n-\frac{\alpha}{k}-1}f^{(n)}(t)dt\Big]$$
$$=\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),m\varphi(a))}{\eta(\varphi(b),m\varphi(a))} \tag{2.2}$$
$$\times\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),m\varphi(a)))d[g(t)]$$
$$-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),m\varphi(b))}{\eta(\varphi(b),m\varphi(a))} \tag{2.3}$$
$$\times\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),m\varphi(b)))d[g(t)].$$

Throughout this paper we denote

$$I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b):=\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x),m\varphi(a))}{\eta(\varphi(b),m\varphi(a))} \tag{2.4}$$

$$\times \int_0^1 g^{n-\frac{\alpha}{k}}(t) f^{(n+1)}(m\varphi(a) + g(t)\eta(\varphi(x), m\varphi(a)))d[g(t)]$$

$$-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), m\varphi(b))}{\eta(\varphi(b), m\varphi(a))}$$

$$\times \int_0^1 g^{n-\frac{\alpha}{k}}(t) f^{(n+1)}(m\varphi(b) + g(t)\eta(\varphi(x), m\varphi(b)))d[g(t)].$$

Proof. Integrating by parts, we get

$$I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b) = \frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), m\varphi(a))}{\eta(\varphi(b), m\varphi(a))}$$

$$\times \Big[\frac{t^{n-\frac{\alpha}{k}} f^{(n)}(m\varphi(a) + t\eta(\varphi(x), m\varphi(a)))}{\eta(\varphi(x), m\varphi(a))}\Big|_{g(0)}^{g(1)}$$

$$-\frac{n-\frac{\alpha}{k}}{\eta(\varphi(x), m\varphi(a))}\int_{g(0)}^{g(1)} t^{n-\frac{\alpha}{k}-1} f^{(n)}(m\varphi(a) + t\eta(\varphi(x), m\varphi(a)))dt\Big]$$

$$-\frac{\eta^{n-\frac{\alpha}{k}+1}(\varphi(x), m\varphi(b))}{\eta(\varphi(b), m\varphi(a))} \times \Big[\frac{t^{n-\frac{\alpha}{k}} f^{(n)}(m\varphi(b) + t\eta(\varphi(x), m\varphi(b)))}{\eta(\varphi(x), m\varphi(b))}\Big|_{g(0)}^{g(1)}$$

$$-\frac{n-\frac{\alpha}{k}}{\eta(\varphi(x), m\varphi(b))}\int_{g(0)}^{g(1)} t^{n-\frac{\alpha}{k}-1} f^{(n)}(m\varphi(b) + t\eta(\varphi(x), m\varphi(b)))dt\Big]$$

$$= \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x), m\varphi(a))}{\eta(\varphi(b), m\varphi(a))} \times [g^{n-\frac{\alpha}{k}}(1) f^{(n)}(m\varphi(a) + g(1)\eta(\varphi(x), m\varphi(a)))$$

$$-g^{n-\frac{\alpha}{k}}(0) f^{(n)}(m\varphi(a) + g(0)\eta(\varphi(x), m\varphi(a)))]$$

$$-\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x), m\varphi(b))}{\eta(\varphi(b), m\varphi(a))} \times [g^{n-\frac{\alpha}{k}}(1) f^{(n)}(m\varphi(b) + g(1)\eta(\varphi(x), m\varphi(b)))$$

$$-g^{n-\frac{\alpha}{k}}(0) f^{(n)}(m\varphi(b) + g(0)\eta(\varphi(x), m\varphi(b)))]$$

$$-\frac{n-\frac{\alpha}{k}}{\eta(\varphi(b), m\varphi(a))} \times \Big[\int_{m\varphi(a)+g(0)\eta(\varphi(x),m\varphi(a))}^{m\varphi(a)+g(1)\eta(\varphi(x),m\varphi(a))} (t - m\varphi(a))^{n-\frac{\alpha}{k}-1} f^{(n)}(t)dt$$

$$-\int_{m\varphi(b)+g(0)\eta(\varphi(x),m\varphi(b))}^{m\varphi(b)+g(1)\eta(\varphi(x),m\varphi(b))} (t - m\varphi(b))^{n-\frac{\alpha}{k}-1} f^{(n)}(t)dt\Big].$$

So, the proof of this lemma is completed. □

Remark 2.6. Under the same conditions as in Lemma 2.5 for $g(t) = t$, we get

$$\begin{aligned} &I_{f,\eta,\varphi}(x;\alpha,k,n,m,a,b) \\ &= \frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),m\varphi(a))f^{(n)}(m\varphi(a)+\eta(\varphi(x),m\varphi(a)))}{\eta(\varphi(b),m\varphi(a))} \\ &-\frac{\eta^{n-\frac{\alpha}{k}}(\varphi(x),m\varphi(b))f^{(n)}(m\varphi(b)+\eta(\varphi(x),m\varphi(b)))}{\eta(\varphi(b),m\varphi(a))} \\ &+(-1)^{n+1}\frac{(nk-\alpha)\Gamma_k\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b),m\varphi(a))} \\ &\times[\,{}^{c}D^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(x),m\varphi(a)))^-}f(m\varphi(a)) - {}^{c}D^{\alpha,k}_{(m\varphi(b)+\eta(\varphi(x),m\varphi(b)))^-}f(m\varphi(b))]. \end{aligned} \tag{2.5}$$

By using Lemma 2.5, one can extend to the following results.

Theorem 2.7. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, where $p, q > -1$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ a differentiable function. Assume that $f : K \longrightarrow (0,+\infty)$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a)+\eta(\varphi(b), m\varphi(a))]$, where $\eta(\varphi(b), m\varphi(a)) > 0$. If $0 < r \leq 1$ and $f^{(n+1)}$ is beta (r, g)-preinvex function on $[m\varphi(a), m\varphi(a)+\eta(\varphi(b), m\varphi(a))]$, then the following inequality for Caputo k-fractional derivatives holds:*

$$\begin{aligned} &|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)| \leq \frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \\ &\times\Big\{m\left(f^{(n+1)}(\varphi(a))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{p}{r}+1,\frac{q}{r}+1\right) \\ &+\left(f^{(n+1)}(\varphi(x))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{q}{r}+1,\frac{p}{r}+1\right)\Big\}^{\frac{1}{r}} \\ &+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \\ &\times\Big\{m\left(f^{(n+1)}(\varphi(b))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{p}{r}+1,\frac{q}{r}+1\right) \\ &+\left(f^{(n+1)}(\varphi(x))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{q}{r}+1,\frac{p}{r}+1\right)\Big\}^{\frac{1}{r}}, \end{aligned} \tag{2.6}$$

where

$$B_{g(x)}(a,b) := \int_{g(0)}^{g(x)} t^{a-1}(1-t)^{b-1}dt.$$

Proof. From Lemma 2.5, beta (r,g)-preinvexity of $f^{(n+1)}$, Minkowski's inequality and properties of the modulus, we have

$$
\begin{aligned}
|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)| &\leq \frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),m\varphi(a))|} \\
&\times \int_0^1 g^{n-\frac{\alpha}{k}}(t) f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),m\varphi(a)))d[g(t)] \\
&+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),m\varphi(a))|} \\
&\times \int_0^1 g^{n-\frac{\alpha}{k}}(t) f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),m\varphi(b)))d[g(t)] \\
&\leq \frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \times \Bigg(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\Big[mg^p(t)(1-g(t))^q \left(f^{(n+1)}(\varphi(a))\right)^r \\
&+g^q(t)(1-g(t))^p \left(f^{(n+1)}(\varphi(x))\right)^r\Big]^{\frac{1}{r}} d[g(t)]\Bigg) \\
&+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \times \Bigg(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\Big[mg^p(t)(1-g(t))^q \left(f^{(n+1)}(\varphi(b))\right)^r \\
&+g^q(t)(1-g(t))^p \left(f^{(n+1)}(\varphi(x))\right)^r\Big]^{\frac{1}{r}} d[g(t)]\Bigg) \\
&\leq \frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \\
&\times\Bigg\{ \left(\int_0^1 m^{\frac{1}{r}} g^{n-\frac{\alpha}{k}+\frac{p}{r}}(t)(1-g(t))^{\frac{q}{r}} \left(f^{(n+1)}(\varphi(a))\right) d[g(t)]\right)^r \\
&+\left(\int_0^1 g^{n-\frac{\alpha}{k}+\frac{q}{r}}(t)(1-g(t))^{\frac{p}{r}} \left(f^{(n+1)}(\varphi(x))\right) d[g(t)]\right)^r \Bigg\}^{\frac{1}{r}} \\
&+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \\
&\times\Bigg\{ \left(\int_0^1 m^{\frac{1}{r}} g^{n-\frac{\alpha}{k}+\frac{p}{r}}(t)(1-g(t))^{\frac{q}{r}} \left(f^{(n+1)}(\varphi(b))\right) d[g(t)]\right)^r \\
&+\left(\int_0^1 g^{n-\frac{\alpha}{k}+\frac{q}{r}}(t)(1-g(t))^{\frac{p}{r}} \left(f^{(n+1)}(\varphi(x))\right) d[g(t)]\right)^r \Bigg\}^{\frac{1}{r}} \\
&= \frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \\
&\times\Bigg\{ m\left(f^{(n+1)}(\varphi(a))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{p}{r}+1,\frac{q}{r}+1\right) \\
&+\left(f^{(n+1)}(\varphi(x))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{q}{r}+1,\frac{p}{r}+1\right)\Bigg\}^{\frac{1}{r}}
\end{aligned}
$$

$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}$$
$$\times\Big\{m\left(f^{(n+1)}(\varphi(b))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{p}{r}+1,\frac{q}{r}+1\right)$$
$$+\left(f^{(n+1)}(\varphi(x))\right)^r B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{q}{r}+1,\frac{p}{r}+1\right)\Big\}^{\frac{1}{r}}.$$

So, the proof of this theorem is completed. □

Corollary 2.8. *Under the same conditions as in Theorem 2.7 for $g(t)=t$, we get*

$$|I_{f,\eta,\varphi}(x;\alpha,k,n,m,a,b)|\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))} \tag{2.7}$$
$$\times\Big\{m\left(f^{(n+1)}(\varphi(a))\right)^r \beta^r\left(n-\frac{\alpha}{k}+\frac{p}{r}+1,\frac{q}{r}+1\right)$$
$$+\left(f^{(n+1)}(\varphi(x))\right)^r \beta^r\left(n-\frac{\alpha}{k}+\frac{q}{r}+1,\frac{p}{r}+1\right)\Big\}^{\frac{1}{r}}$$
$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}$$
$$\times\Big\{m\left(f^{(n+1)}(\varphi(b))\right)^r \beta^r\left(n-\frac{\alpha}{k}+\frac{p}{r}+1,\frac{q}{r}+1\right)$$
$$+\left(f^{(n+1)}(\varphi(x))\right)^r \beta^r\left(n-\frac{\alpha}{k}+\frac{q}{r}+1,\frac{p}{r}+1\right)\Big\}^{\frac{1}{r}}.$$

Corollary 2.9. *Under the same conditions as in Theorem 2.7, if we choose $p=0,q=s,m=k=r=1$, $\eta(\varphi(y),m\varphi(x))=\varphi(y)-m\varphi(x)$, $\varphi(x)=x$, $g(t)=t$ and $f^{(n+1)}\leq M$, we get the following inequality for Caputo fractional derivatives:*

$$\Big|\Big[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\Big]f^{(n)}(x)$$
$$+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\big[{}^cD^{\alpha}_{x-}f(a)-{}^cD^{\alpha}_{x-}f(b)\big]\Big|$$
$$\leq M\left(\beta(n-\alpha+1,s+1)+\frac{1}{n+s-\alpha+1}\right) \tag{2.8}$$
$$\times\Big[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\Big].$$

Theorem 2.10. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, where $\gamma, s > -1$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0, 1] \longrightarrow [0, 1]$ a differentiable function. Assume that $f : K \longrightarrow (0, +\infty)$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a) + \eta(\varphi(b), m\varphi(a))]$, where $\eta(\varphi(b), m\varphi(a)) > 0$. If $0 < r \leq 1$ and $\left(f^{(n+1)}\right)^q$ is beta (r, g)-preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), m\varphi(a))]$, $q > 1$, $p^{-1} + q^{-1} = 1$, then the following inequality for Caputo k-fractional derivatives holds:*

$$
\begin{aligned}
&|I_{f,g,\eta,\varphi}(x; \alpha, k, n, m, a, b)| \\
&\leq \left(\frac{g^{(n-\frac{\alpha}{k})p+1}(1) - g^{(n-\frac{\alpha}{k})p+1}(0)}{(n - \frac{\alpha}{k})p + 1} \right)^{\frac{1}{p}} \frac{1}{\eta(\varphi(b), m\varphi(a))} \qquad (2.9)\\
&\times \Big\{ |\eta(\varphi(x), m\varphi(a))|^{n-\frac{\alpha}{k}+1} \Big[m \left(f^{(n+1)}(\varphi(a)) \right)^{rq} B^r_{g(1)} \left(\frac{\gamma}{r} + 1, \frac{s}{r} + 1 \right) \\
&\quad + \left(f^{(n+1)}(\varphi(x)) \right)^{rq} B^r_{g(1)} \left(\frac{s}{r} + 1, \frac{\gamma}{r} + 1 \right) \Big]^{\frac{1}{rq}} \\
&+ |\eta(\varphi(x), m\varphi(b))|^{n-\frac{\alpha}{k}+1} \Big[m \left(f^{(n+1)}(\varphi(b)) \right)^{rq} B^r_{g(1)} \left(\frac{\gamma}{r} + 1, \frac{s}{r} + 1 \right) \\
&\quad + \left(f^{(n+1)}(\varphi(x)) \right)^{rq} B^r_{g(1)} \left(\frac{s}{r} + 1, \frac{\gamma}{r} + 1 \right) \Big]^{\frac{1}{rq}} \Big\}.
\end{aligned}
$$

Proof. From Lemma 2.5, beta (r, g)-preinvexity of $\left(f^{(n+1)}\right)^q$, Hölder's inequality, Minkowski's inequality and properties of the modulus, we have

$$
\begin{aligned}
|I_{f,g,\eta,\varphi}(x; \alpha, k, n, m, a, b)| &\leq \frac{|\eta(\varphi(x), m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b), m\varphi(a))|} \\
&\times \int_0^1 g^{n-\frac{\alpha}{k}}(t) f^{(n+1)}(m\varphi(a) + g(t)\eta(\varphi(x), m\varphi(a))) d[g(t)] \\
&+ \frac{|\eta(\varphi(x), m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b), m\varphi(a))|} \\
&\times \int_0^1 g^{n-\frac{\alpha}{k}}(t) f^{(n+1)}(m\varphi(b) + g(t)\eta(\varphi(x), m\varphi(b))) d[g(t)] \\
&\leq \frac{|\eta(\varphi(x), m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), m\varphi(a))} \left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t) d[g(t)] \right)^{\frac{1}{p}} \\
&\times \left(\int_0^1 \left(f^{(n+1)}(m\varphi(a) + g(t)\eta(\varphi(x), m\varphi(a))) \right)^q d[g(t)] \right)^{\frac{1}{q}} \\
&+ \frac{|\eta(\varphi(x), m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b), m\varphi(a))} \left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t) d[g(t)] \right)^{\frac{1}{p}}
\end{aligned}
$$

$$\times\left(\int_0^1\left(f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),m\varphi(b)))\right)^q d[g(t)]\right)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\Big(\int_0^1[mg^{\gamma}(t)(1-g(t))^s\left(f^{(n+1)}(\varphi(a))\right)^{rq}$$

$$+g^s(t)(1-g(t))^{\gamma}\left(f^{(n+1)}(\varphi(x))\right)^{rq}]^{\frac{1}{r}}d[g(t)]\Big)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\Big(\int_0^1[mg^{\gamma}(t)(1-g(t))^s\left(f^{(n+1)}(\varphi(b))\right)^{rq}$$

$$+g^s(t)(1-g(t))^{\gamma}\left(f^{(n+1)}(\varphi(x))\right)^{rq}]^{\frac{1}{r}}d[g(t)]\Big)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\Big\{\left(\int_0^1 m^{\frac{1}{r}}g^{\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(a))\right)^q d[g(t)]\right)^r$$

$$+\left(\int_0^1 g^{\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}\left(f^{(n+1)}(\varphi(x))\right)^q d[g(t)]\right)^r\Big\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{\left(n-\frac{\alpha}{k}\right)p}(t)d[g(t)]\right)^{\frac{1}{p}}$$

$$\times\Big\{\left(\int_0^1 m^{\frac{1}{r}}g^{\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(b))\right)^q d[g(t)]\right)^r$$

$$+\left(\int_0^1 g^{\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}\left(f^{(n+1)}(\varphi(x))\right)^q d[g(t)]\right)^r\Big\}^{\frac{1}{rq}}$$

$$=\left(\frac{g^{(n-\frac{\alpha}{k})p+1}(1)-g^{(n-\frac{\alpha}{k})p+1}(0)}{(n-\frac{\alpha}{k})p+1}\right)^{\frac{1}{p}}\frac{1}{\eta(\varphi(b),m\varphi(a))}$$

$$\times\Big\{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq}B^r_{g(1)}\left(\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}B^r_{g(1)}\left(\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq}B^r_{g(1)}\left(\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq} B_{g(1)}^{r}\left(\frac{s}{r}+1, \frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}\Big\}.$$

So, the proof of this theorem is completed. □

Corollary 2.11. *Under the same conditions as in Theorem 2.10 for $g(t) = t$, we get*

$$|I_{f,\eta,\varphi}(x;\alpha,k,n,m,a,b)| \leq \left(\frac{1}{(n-\frac{\alpha}{k})p+1}\right)^{\frac{1}{p}} \frac{\beta^{\frac{1}{q}}\left(\frac{s}{r}+1, \frac{\gamma}{r}+1\right)}{\eta(\varphi(b), m\varphi(a))} \tag{2.10}$$

$$\times\Big\{|\eta(\varphi(x), m\varphi(a))|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq}+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x), m\varphi(b))|^{n-\frac{\alpha}{k}+1}\Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq}+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\Big]^{\frac{1}{rq}}\Big\}.$$

Corollary 2.12. *Under the same conditions as in Theorem 2.10, if we choose $\gamma = 0, m = k = r = 1$, $\eta(\varphi(y), m\varphi(x)) = \varphi(y) - m\varphi(x)$, $\varphi(x) = x$, $g(t) = t$ and $f^{(n+1)} \leq M$, we get the following inequality for Caputo fractional derivatives:*

$$\Big|\Big[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\Big]f^{(n)}(x)$$

$$+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\left[{}^{c}D_{x-}^{\alpha}f(a) - {}^{c}D_{x-}^{\alpha}f(b)\right]\Big|$$

$$\leq \frac{M}{((n-\alpha)p+1)^{\frac{1}{p}}}\left(\frac{2}{s+1}\right)^{\frac{1}{q}}\Big[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\Big]. \tag{2.11}$$

Theorem 2.13. *Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. Suppose $K \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, where $\gamma, s > -1$. Let $\varphi : I \longrightarrow K$ be a continuous function and $g : [0,1] \longrightarrow [0,1]$ a differentiable function. Assume that $f : K \longrightarrow (0,+\infty)$ is a function on K° such that $f \in C^{n+1}[m\varphi(a), m\varphi(a) + \eta(\varphi(b), m\varphi(a))]$, where $\eta(\varphi(b), m\varphi(a)) > 0$. If $0 < r \leq 1$ and $\left(f^{(n+1)}\right)^{q}$ is beta (r,g)-preinvex function on $[m\varphi(a), m\varphi(a) + \eta(\varphi(b), m\varphi(a))]$, $q \geq 1$, then the following inequality for Caputo k-fractional derivatives holds:*

$$|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)|$$

$$\leq \left(\frac{g^{n-\frac{\alpha}{k}+1}(1)-g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b), m\varphi(a))} \tag{2.12}$$

$$\times\Big\{|\eta(\varphi(x), m\varphi(a))|^{n-\frac{\alpha}{k}+1}$$

$$\times\Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq}B^{r}_{g(1)}\left(n-\frac{\alpha}{k}+\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$
$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}B^{r}_{g(1)}\left(n-\frac{\alpha}{k}+\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}$$
$$+|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}$$
$$\times\Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq}B^{r}_{g(1)}\left(n-\frac{\alpha}{k}+\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$
$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}B^{r}_{g(1)}\left(n-\frac{\alpha}{k}+\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}\Big\}.$$

Proof. From Lemma 2.5, beta (r,g)-preinvexity of $\left(f^{(n+1)}\right)^q$, the well-known power mean inequality, Minkowski's inequality and properties of the modulus, we have

$$|I_{f,g,\eta,\varphi}(x;\alpha,k,n,m,a,b)|\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),m\varphi(a))|}$$
$$\times\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),m\varphi(a)))d[g(t)]$$
$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{|\eta(\varphi(b),m\varphi(a))|}$$
$$\times\int_0^1 g^{n-\frac{\alpha}{k}}(t)f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),m\varphi(b)))d[g(t)]$$
$$\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\left(f^{(n+1)}(m\varphi(a)+g(t)\eta(\varphi(x),m\varphi(a)))\right)^q d[g(t)]\right)^{\frac{1}{q}}$$
$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)\left(f^{(n+1)}(m\varphi(b)+g(t)\eta(\varphi(x),m\varphi(b)))\right)^q d[g(t)]\right)^{\frac{1}{q}}$$
$$\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}$$
$$\times\Big(\int_0^1 g^{n-\frac{\alpha}{k}}(t)[mg^{\gamma}(t)(1-g(t))^s\left(f^{(n+1)}(\varphi(a))\right)^{rq}$$
$$+g^s(t)(1-g(t))^{\gamma}\left(f^{(n+1)}(\varphi(x))\right)^{rq}]^{\frac{1}{r}}d[g(t)]\Big)^{\frac{1}{q}}$$

$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Big(\int_0^1 g^{n-\frac{\alpha}{k}}(t)[mg^{\gamma}(t)(1-g(t))^s\left(f^{(n+1)}(\varphi(b))\right)^{rq}$$

$$+g^s(t)(1-g(t))^{\gamma}\left(f^{(n+1)}(\varphi(x))\right)^{rq}]^{\frac{1}{r}}d[g(t)]\Big)^{\frac{1}{q}}$$

$$\leq\frac{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Big\{\left(\int_0^1 m^{\frac{1}{r}}g^{n-\frac{\alpha}{k}+\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(a))\right)^{q}d[g(t)]\right)^{r}$$

$$+\left(\int_0^1 g^{n-\frac{\alpha}{k}+\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}\left(f^{(n+1)}(\varphi(x))\right)^{q}d[g(t)]\right)^{r}\Big\}^{\frac{1}{rq}}$$

$$+\frac{|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}}{\eta(\varphi(b),m\varphi(a))}\left(\int_0^1 g^{n-\frac{\alpha}{k}}(t)d[g(t)]\right)^{1-\frac{1}{q}}$$

$$\times\Big\{\left(\int_0^1 m^{\frac{1}{r}}g^{n-\frac{\alpha}{k}+\frac{\gamma}{r}}(t)(1-g(t))^{\frac{s}{r}}\left(f^{(n+1)}(\varphi(b))\right)^{q}d[g(t)]\right)^{r}$$

$$+\left(\int_0^1 g^{n-\frac{\alpha}{k}+\frac{s}{r}}(t)(1-g(t))^{\frac{\gamma}{r}}\left(f^{(n+1)}(\varphi(x))\right)^{q}d[g(t)]\right)^{r}\Big\}^{\frac{1}{rq}}$$

$$=\left(\frac{g^{n-\frac{\alpha}{k}+1}(1)-g^{n-\frac{\alpha}{k}+1}(0)}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}}\frac{1}{\eta(\varphi(b),m\varphi(a))}$$

$$\times\Big\{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}$$

$$\times\Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq}B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}$$

$$\times\Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq}B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}B^r_{g(1)}\left(n-\frac{\alpha}{k}+\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}\Big\}.$$

So, the proof of this theorem is completed. □

Corollary 2.14. *Under the same conditions as in Theorem 2.13 for $g(t) = t$, we get*

$$|I_{f,\eta,\varphi}(x;\alpha,k,n,m,a,b)| \leq \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \frac{1}{\eta(\varphi(b),m\varphi(a))} \tag{2.13}$$

$$\times\Big\{|\eta(\varphi(x),m\varphi(a))|^{n-\frac{\alpha}{k}+1}$$

$$\times\Big[m\left(f^{(n+1)}(\varphi(a))\right)^{rq}\beta^{r}\left(n-\frac{\alpha}{k}+\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\beta^{r}\left(n-\frac{\alpha}{k}+\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}$$

$$+|\eta(\varphi(x),m\varphi(b))|^{n-\frac{\alpha}{k}+1}$$

$$\times\Big[m\left(f^{(n+1)}(\varphi(b))\right)^{rq}\beta^{r}\left(n-\frac{\alpha}{k}+\frac{\gamma}{r}+1,\frac{s}{r}+1\right)$$

$$+\left(f^{(n+1)}(\varphi(x))\right)^{rq}\beta^{r}\left(n-\frac{\alpha}{k}+\frac{s}{r}+1,\frac{\gamma}{r}+1\right)\Big]^{\frac{1}{rq}}\Big\}.$$

Corollary 2.15. *Under the same conditions as in Theorem 2.13, if we choose $\gamma = 0, m = k = r = 1$, $\eta(\varphi(y), m\varphi(x)) = \varphi(y) - m\varphi(x)$, $\varphi(x) = x$, $g(t) = t$ and $f^{(n+1)} \leq M$, we get the following inequality for Caputo fractional derivatives:*

$$\Big|\Big[\frac{(x-a)^{n-\alpha}-(x-b)^{n-\alpha}}{b-a}\Big]f^{(n)}(x)$$

$$+(-1)^{n+1}\frac{\Gamma(n-\alpha+1)}{b-a}\left[{}^{c}D^{\alpha}_{x-}f(a)-{}^{c}D^{\alpha}_{x-}f(b)\right]\Big|$$

$$\leq M\left(\frac{1}{n-\alpha+1}\right)^{1-\frac{1}{q}}\left(\beta(n-\alpha+1,s+1)+\frac{1}{n+s-\alpha+1}\right)^{\frac{1}{q}} \tag{2.14}$$

$$\times\Big[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{b-a}\Big].$$

Corollary 2.16. *Under the same conditions as in Theorem 2.13 for $q = 1$, we get Theorem 2.7.*

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