Математички Билтен ISSN 0351-336X
36(LXII)
2012(33-46)
Скопје, Македонија

ON FRACTIONAL q-KINETIC EQUATION

MRIDULA GARG AND LATA CHANCHLANI

Abstract. In this paper, we obtain the solution of fractional q-kinetic equation, which involves the Riemann-Liouville fractional q-integral operator. We apply the method of q-Laplace transform and its inverse to obtain the solution in closed form. The solutions of ordinary q-kinetic equation and two more fractional q-kinetic equations are obtained as special cases of our main result. We have also drawn some graphs of the solutions of fractional q-kinetic equations using the software MATHEMATICA.

1. **Introduction**

Kinetic equations describe the continuity of motion of substance and are the basic equations of mathematical physics and natural science. They can be used for quantitative and qualitative description of physical, chemical, biological, social and other processes. They are called "master equations" because mathematical simulation of the evolutionary processes based on the kinetic equation is fruitful and effective. The Liouville equation, the Boltzmann equation, the system of Vlasov's equation, the Fokker-Planck radiation transfer equations, the chain of Bogoljubov's equations, the Vegner equations for the matrix of density and diffraction are important kinetic equations

The study of q-analysis is an old subject, which dates back to the end of the19th century [17]. A detailed account of the work can be seen in the books by Slater [21], Exton [5], Gasper and Rahman [7] and a licentiate thesis by Ernst [4]. The q-analysis has found many applications in such areas as the theory of partitions, combinatorics, exactly solvable models in statistical mechanics, computer algebra etc [3]. Recent developments in the theory of quantum group as given in [6] and [11] have boosted further interests in this old subject. The subject of q-analysis concerns mainly the

---

2000 *Mathematics Subject Classification.* 33D05, 39A13, 33E12, 35A22.
Key words and phrases. Fractional q-kinetic equation, q-Laplace transform, Riemann-Liouville fractional q-integral operator, q-analogue of Mittag-Leffler function.

properties of the so-called q-special functions, which are the extensions of the classical special functions based on a parameter, or the base q. In recent years, mathematicians have reconsidered q-difference equations for their links with other branches of mathematics such as quantum algebras and q-combinatory. The q-difference equations involve a new kind of difference operator, the q-derivative, which can be viewed as a sort of deformation of the ordinary derivative. Solutions of the q-difference equations in one variable have been well studied in terms of the q-hypergeometric series (also called the basic hypergeometric series).

The differential equations with fractional order have recently proved to be valuable tools to the modeling of many physical phenomena ([14], [13], [15], [10]). This is because of the fact that the realistic modeling of a physical phenomenon does not depend only on the instant time, but also on the instant time which can also be successfully achieved by using fractional calculus. Their treatment from the point of view of the q-calculus by using fractional q-calculus operators of Riemann-Liouville type can additionally open new perspective.

The standard kinetic equation is given by

$$\frac{dN}{dt} = -cN\left(t\right), \tag{1}$$

with the initial condition that $N\left(t=0\right) = N_0$ is the number density at time t=0, $c > 0$ is a constant. A detailed discussion of the above equation is given in [12]. The solution of the above standard kinetic equation can be put in the following form:

$$N\left(t\right) = N_0 e^{-ct}. \tag{2}$$

The fractional generalization of the standard kinetic equation (1) is given by Haubold and Mathai [9], in the following form:

$$N\left(t\right) - N_0 = -cD_t^{-\nu} N\left(t\right), \tag{3}$$

where ${}_0D_t^{-\nu}$ is the well-known Riemann-Liouville fractional integral operator [14] defined by

$$D_t^{-\nu} f\left(t\right) = \frac{1}{\Gamma\left(\nu\right)} \int_0^t \left(t-u\right)^{\nu-1} f\left(u\right) du,\, R\left(\nu\right) > 0, \tag{4}$$

with ${}_0D_t^0 f\left(t\right) = f\left(t\right)$.

Saxena, Mathai and Haubold have further studied more generalized forms of fractional kinetic equation in a series of papers ([9], [18], [19], [20]).

2. **Preliminaries**

The following definitions and notations have been taken from the book by Gasper and Rahman [7].

For real or complex a and $0 < |q| < 1$, the q-shifted factorial is defined as:

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \tag{5}$$

where

$$(a;q)_\infty = \prod_{j=0}^{\infty} \left(1 - aq^j\right).$$

Its extension to real α is

$$(a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty}, (\alpha \in R). \tag{6}$$

Also, the q-analogue of the power function $(a-b)^\alpha$is

$$(a-b)_q^\alpha = a^\alpha \left({}^b\!/_a;q\right)_\alpha, (a \neq 0), \tag{7}$$

The q-gamma function is defined by

$$\Gamma_q(\alpha) = \frac{G(q^\alpha)}{G(q)}(1-q)^{1-\alpha} = (1-q)_q^{\alpha-1}(1-q)^{1-\alpha}; R(\alpha) > 0, \tag{8}$$

where $G(q^\alpha) = \frac{1}{(q^\alpha;q)_\infty}$ and $G(q) = \frac{1}{(q;q)_\infty}$.

The q-factorial is defined by

$$[n]_q! = \prod_{k=1}^{n} [k]_q \text{ and } [0]_q! = 1, \tag{9}$$

where q-number $[k]_q = \frac{1-q^k}{1-q},\ q \neq 1$.

The q-beta function is defined by

$$\begin{array}{l} B_q(a,b) = \frac{\Gamma_q(a)\Gamma_q(b)}{\Gamma_q(a+b)} = \int_0^1 z^{a-1}(1-qz)_q^{b-1} d_q z, \\ Re(a) > 0,\ b \neq 0, -1, -2, ... \end{array} \tag{10}$$

where q-integral is defined as follows:

$$\int_0^a f(z)\, d_q z = a(1-q)\sum_{n=0}^{\infty} q^n f(aq^n). \tag{11}$$

The q-extensions of the exponential function are given by

$$E_q^z = {}_0\phi_0\left(-;-;q,-z\right) = \sum_{n=0}^{\infty} \frac{q^{\binom{n}{2}} z^n}{\left(q;q\right)_n} = \left(-z;q\right)_\infty, \tag{12}$$

$$e_q^z = {}_1\phi_0\left(0;-;q,z\right) = \sum_{n=0}^{\infty} \frac{z^n}{\left(q;q\right)_n} = \frac{1}{\left(z;q\right)_\infty}, \ \ |z| < 1. \tag{13}$$

We define q-analogues of generalized Mittag-Leffler type functions $E_{\alpha,\beta}^{\gamma}\left(z\right)$ and $E_{\alpha,\beta}\left(z\right)$ defined by Prabhaker [16] and Wiman [22] respectively as follows

$$E_{\alpha,\beta}^{\gamma}\left(z;q\right) = \sum_{n=0}^{\infty} \frac{\left(q^{\gamma};q\right)_n z^n}{\Gamma_q\left(\alpha n+\beta\right)\left[n\right]_q!}, \tag{14}$$

and

$$E_{\alpha,\beta}\left(z;q\right) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma_q\left(\alpha n+\beta\right)}, \tag{15}$$

for $\alpha > 0; \beta, \gamma, z \in C$.

q-Laplace transform

Hahn ([8], see also [1]) defined the q-analogue of the well-known classical Laplace transform by means of the following q-integral

$${}_qL_s\left\{f\left(t\right)\right\} = \frac{1}{\left(1-q\right)} \int_0^{s^{-1}} E_q^{-qst} f\left(t\right) d_qt, \tag{16}$$

where E_q^z is the q-exponential defined in (12).

Some results for q-Laplace transform are given as follows:
q-Laplace convolution theorem

$${}_qL_s\left(f *_q g\right)\left(s\right) = {}_qL_s\left\{f\left(t\right)\right\} {}_qL_s\left\{g\left(t\right)\right\}, \tag{17}$$

where $f *_q g$ is the q-convolution of two analytic functions $f\left(t\right)$ and $g\left(t\right)$ defined as

$$f\left(t\right) *_q g\left(t\right) = \frac{1}{\left(1-q\right)} \int_0^t f\left(u\right) g\left[t-qu\right] d_qu. \tag{18}$$

where for the function $g\left(t\right) = \sum_{n=0}^{\infty} a_n t^n$,

$$g\left[t-qu\right] = \sum_{n=0}^{\infty} a_n (t-qu)(t-q^2u)...(t-q^nu). \tag{19}$$

q-Laplace transform of power function

$$ {}_qL_s\left\{t^{\nu}\right\}=\frac{(q;q)_{\nu}}{s^{\nu+1}}=\frac{(1-q)^{\nu}\,\Gamma_q\left(\nu+1\right)}{s^{\nu+1}};R(\nu)>0, \tag{20}$$

q-Laplace transform of Riemann-Liouville fractional q-integral operator

$$ {}_qL_s\left\{I_q^{\alpha}f\left(t\right)\right\}=\left(\frac{(1-q)}{s}\right)^{\alpha}{}_qL_s\left\{f\left(t\right)\right\};R\left(\alpha\right)>0. \tag{21}$$

where the Riemann-Liouville fractional q-integral operator [2], is defined as

$$ I_q^{\alpha}f\left(t\right)=\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^t\left(t-zq\right)_q^{\alpha-1}f\left(z\right)d_qz;R\left(\alpha\right)>0. \tag{22}$$

Proof. By using (22), we get

$$ {}_qL_s\left\{I_q^{\alpha}f\left(t\right)\right\}={}_qL_s\left\{\frac{1}{\Gamma_q\left(\alpha\right)}\int_0^t\left(t-qu\right)_q^{\alpha-1}f\left(u\right)d_qu\right\}.$$

In view of q-convolution (17), we have

$$ {}_qL_s\left\{I_q^{\alpha}f\left(t\right)\right\}=\frac{1}{\Gamma_q\left(\alpha\right)}{}_qL_s\left\{t^{\alpha-1}\right\}{}_qL_s\left\{f\left(t\right)\right\}. \tag{23}$$

Hence by using (20), we get the required result (21).

3. **Fractional q-kinetic equation**

Theorem. For $c>0,\nu>0,0<|q|<1$ consider the following fractional q-kinetic equation

$$ N_q\left(t\right)-N_0f_q\left(t\right)=-cI_q^{\nu}N_q\left(t\right), \tag{24}$$

where $f_q\left(t\right)$ is any q-integrable function over any finite interval $[0,b]$ and I_q^{ν} is the Riemann-Liouville fractional q-integral operator defined by (22), is given by

$$ N_q\left(t\right)=N_0\int_0^t f_q\left[t-qu\right]u^{-1}E_{\nu,0}\left(-cu^{\nu};q\right)d_qu. \tag{25}$$

Proof. We take q-Laplace transform of equation (24) and use the result (21), to get

$$ N_q^*\left(s\right)=\frac{N_0f_q^*\left(s\right)}{\left[1+\frac{c(1-q)^{\nu}}{s^{\nu}}\right]}, \tag{26}$$

where $f_q^*\left(s\right)$ and $N_q^*\left(s\right)$ are q-Laplace transforms of $f\left(t\right)$ and $N\left(t\right)$ respectively.

Now we obtain ${}_qL_s^{-1}$ of right side of (26). For this we write

$$ {}_qL_s^{-1}\left\{\frac{1}{\left[1+\frac{c(1-q)^{\nu}}{s^{\nu}}\right]}\right\} = {}_qL_s^{-1}\left\{\sum_{r=0}^{\infty}\frac{(-1)^r\, c^r\,(1-q)^{\nu r}}{s^{\nu r}}\right\} $$

$$ = \sum_{r=0}^{\infty}(-1)^r\, c^r\,(1-q)^{\nu r}\, {}_qL_s^{-1}\left\{s^{-\nu r}\right\}. $$

On using the result (20) and the definition (15) of q-analogue of generalized Mittag-Leffler function, we arrive at

$$ {}_qL_s^{-1}\left\{\frac{1}{\left[1+\frac{c(1-q)^{\nu}}{s^{\nu}}\right]}\right\} = (1-q)\, t^{-1} E_{\nu,0}\left(-ct^{\nu};q\right). \tag{27} $$

If we now take the inverse q-Laplace transform of (26), apply the q-Laplace convolution theorem given by (17), we easily arrive at the desired result (25).

4. Special cases

We now obtain solutions of some other fractional q-kinetic equations and ordinary q-kinetic equation by specializing parameters involved in our main theorem.

1. If we set $f_q(t) = t^{\mu-1}$ and apply the definition (10), we get the following result, which is a q-analogue of the result established by Saxena *et al.* [19].

Corollary 1. If $c > 0$, $\nu > 0$, $\mu > 0$, $0 < |q| < 1$ the solution of the q-fractional kinetic equation

$$ N_q(t) - N_0 t^{\mu-1} = -cI_q^{\nu} N_q(t), \tag{28} $$

is given by

$$ N_q(t) = N_0 \Gamma_q(\mu)\, t^{\mu-1} E_{\nu,\mu}\left(-ct^{\nu};q\right). \tag{29} $$

The results of computations of the solution (29), with different values of μ, ν and q are shown in Figures 1 to 9. All the plots are drawn by using the MATHEMATICA for fixed $N_0 = 1$ and $c = .086625$, which is a rate constant of a particular homogeneous gaseous reaction. The graphs are drawn between t and $N_q(t)$. In Figures 1 to 3, we fix up the values of μ, ν and draw graphs for different values of q. In Figures 4 to 6, we fix up the values of ν, q and draw graphs for different values of μ. In Figures 7 to 9, we fix up the values of q, μ and draw graphs for different values of ν.

2. If we take $f_q(t) = t^{\mu-1}E^{\gamma}_{\nu,\mu}[-ct^{\nu};q]$, we get q-analogue of a result given by Saxena *et al.* [20].

Corollary 2. If $c > 0$, $\nu > 0$, $0 < |q| < 1$ then the solution of the fractional q-kinetic equation

$$N_q(t) - N_0 t^{\mu-1} E^{\gamma}_{\nu,\mu}[-ct^{\nu};q] = -cI^{\nu}_q N(t), \tag{30}$$

is given by

$$N_q(t) = N_0 t^{\mu-1}.\sum_{r=0}^{\infty} \frac{(q^{\gamma};q)_r (-ct^{\nu})^r}{[r]_q!} E_{\nu,\mu+\nu r}(-ct^{\nu};q). \tag{31}$$

Proof. By substituting $f_q(t) = t^{\mu-1}E^{\gamma}_{\nu,\mu}(-ct^{\nu};q)$ in the solution (25) of main theorem, we arrive at

$$N_q(t) = N_0 \int_0^t (t-qu)_q^{\mu-1} E^{\gamma}_{\nu,\mu}(-c(t-qu)^{\nu};q)\, u^{-1} E_{\nu,0}(-cu^{\nu};q)\, d_q u. \tag{32}$$

Using definitions of q-analogues of generalized Mittag-Leffler functions defined by (14) and (15), we get

$$N_q(t) = N_0 \int_0^t (t-qu)_q^{\mu-1} \sum_{r=0}^{\infty} \frac{(q^{\gamma};q)_r (-c(t-qu)^{\nu})^r}{\Gamma_q(\nu r+\mu)[r]_q!} u^{-1} \sum_{n=0}^{\infty} \frac{(-cu^{\nu})^n}{\Gamma_q(\nu n)} d_q u.$$

On interchanging the order of integration and summations and using q-beta integral (10) we get

$$= N_0 t^{\mu-1} \sum_{r=0}^{\infty} \frac{(q^{\gamma};q)_r (-ct^{\nu})^r}{[r]_q!} \sum_{n=0}^{\infty} \frac{(-ct^{\nu})^n}{\Gamma_q(\nu n+\mu+\nu r)}.$$

Finally, in view of the definition (15), we arrive at the desired result (31).

The results of computations of the solution (31), with different values of μ, ν, γ and q are shown in Figures 10 to 13. All the plots are drawn by using the MATHEMATICA for fixed $N_0 = 1$ and $c = .086625$, which is a rate constant of a particular homogeneous gaseous reaction. The graphs are drawn between t and $N_q(t)$. In Figure 10, we fix up the values of q, ν, μ and draw graphs for different values of γ. In Figure 11, we fix up the values of γ, ν, q and draw graphs for different values of μ. In Figure 12, we fix up the values of μ, γ, q and draw graphs for different values of ν. In Figure 13, we fix up the values of μ, γ, ν and draw graphs for different values of q.

3. If we take $\nu = 1$ in our theorem, the fractional equation converts into ordinary q-kinetic equation as we get the following result:

Corollary 3. For $c > 0$, $\nu > 0$, $0 < |q| < 1$ the solution of the fractional kinetic equation

$$N_q(t) - N_0 f_q(t) = -cI_q N_q(t), \tag{33}$$

where $f_q(t)$ is any q-integrable function over any finite interval $[0, b]$ and I_q is the q-integral operator, is given by

$$N_q(t) = -cN_0 \int_0^t f_q[t - qu]\, e_q^{(-(1-q)u)} d_q u. \tag{34}$$

4. If we take $q \to 1$ in our theorem, we arrive at the problem considered by Saxena *et al.* [18] and also obtain the solutions of the problems considered by Haubold & Mathai [9] and Saxena *et al.* ([19], [20]) as special cases.

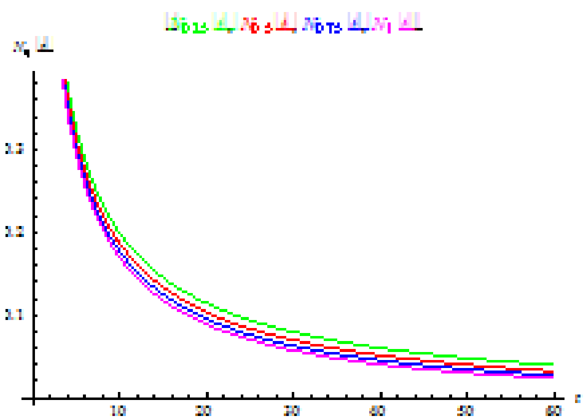

Fig.1 $\mu = .5$, $\nu = .6$ and $q = \{0.25,\ 0.5,\ 0.75,\ 1\}$

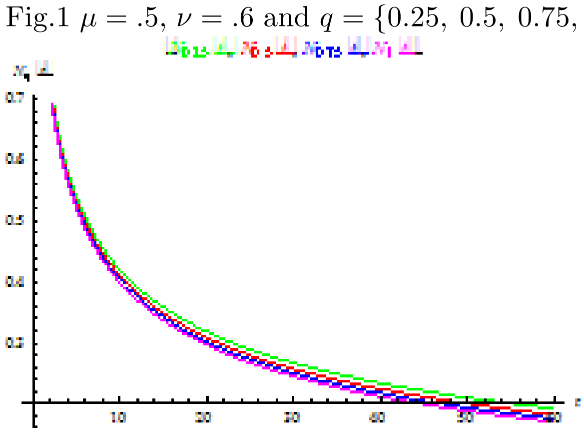

Fig.2 $\mu = .75$, $\nu = .5$ and $q = \{0.25,\ 0.5,\ 0.75,\ 1\}$

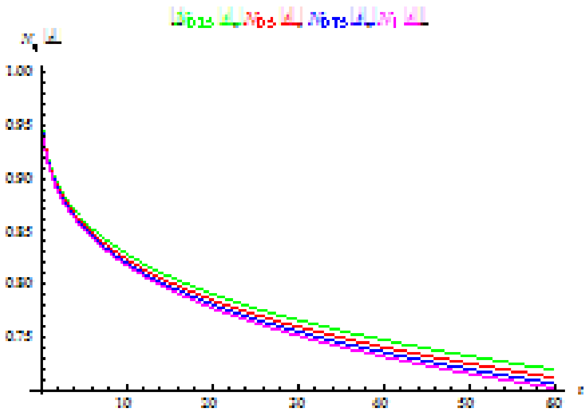

Fig.3 $\mu = 1$, $\nu = .35$ and $q = \{0.25,\ 0.5,\ 0.75,\ 1\}$

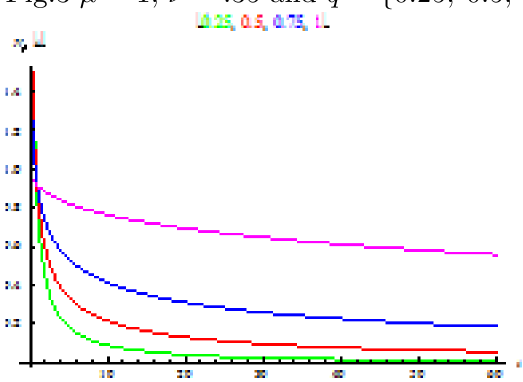

Fig.4 $\nu = .5$, $q = .5$ and $\mu = \{0.25,\ 0.5,\ 0.75,\ 1\}$

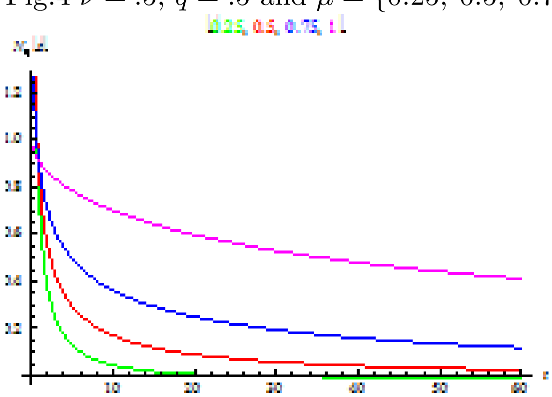

Fig.5 $\nu = .6$, $q = 1$ and $\mu = \{0.25,\ 0.5,\ 0.75,\ 1\}$

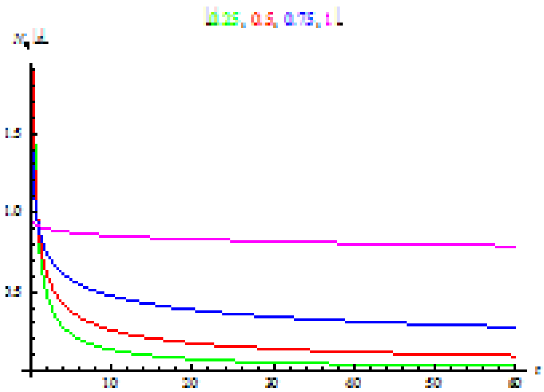

Fig.6 $\nu = .25$, $q = .75$ and $\mu = \{0.25,\ 0.5,\ 0.75,\ 1\}$

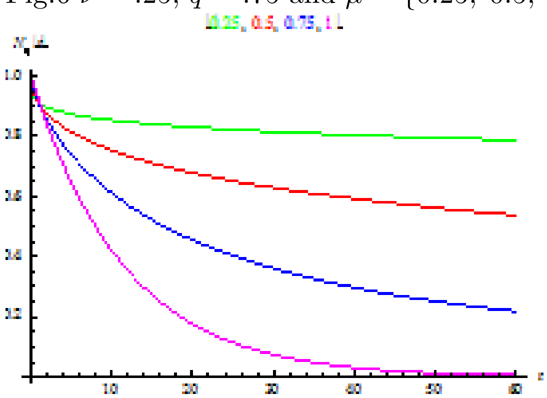

Fig.7 $\mu = 1$, $q = 1$ and $\nu = \{0.25,\ 0.5,\ 0.75,\ 1\}$

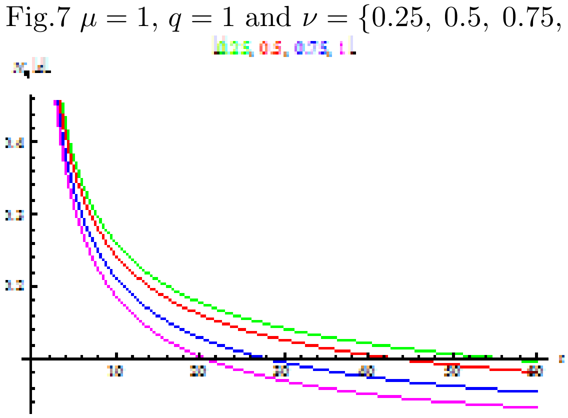

Fig.8 $\mu = 0.5$, $q = 0.5$ and $\nu = \{0.1,\ 0.25,\ 0.35,\ 0.5\}$

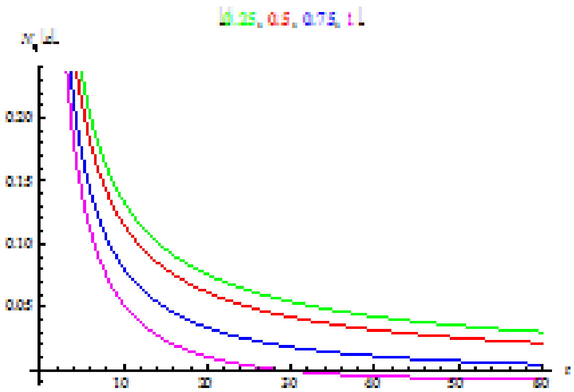

Fig.9 $\mu = 0.25$, $q = 0.75$ and $\nu = \{0.25,\ 0.35,\ 0.5,\ 0.6\}$

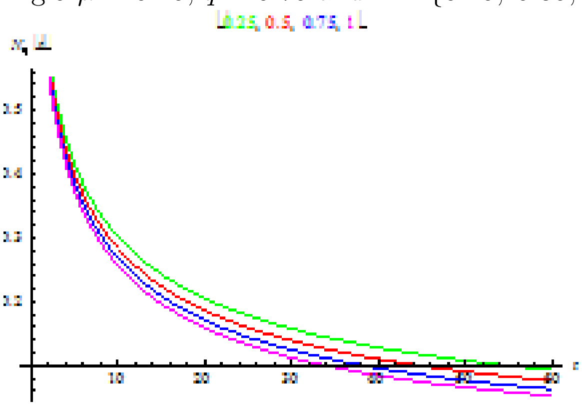

Fig.10 $\mu = .75$, $\nu = .6$, $q = .5$ and $\gamma = \{0.25,\ 0.5,\ 0.75,\ 1\}$

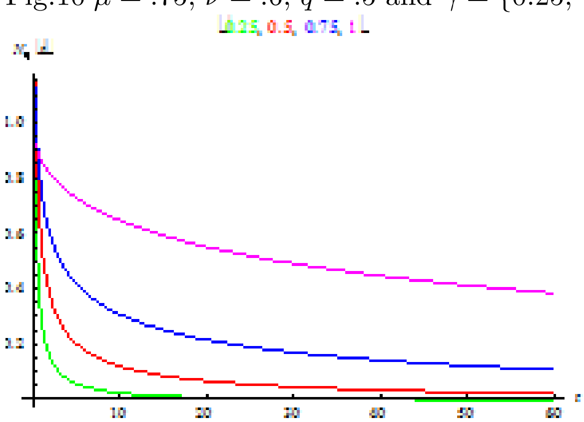

Fig.11 $\gamma = .75$, $\nu = .5$, $q = .25$ and $\mu = \{0.25,\ 0.5,\ 0.75,\ 1\}$

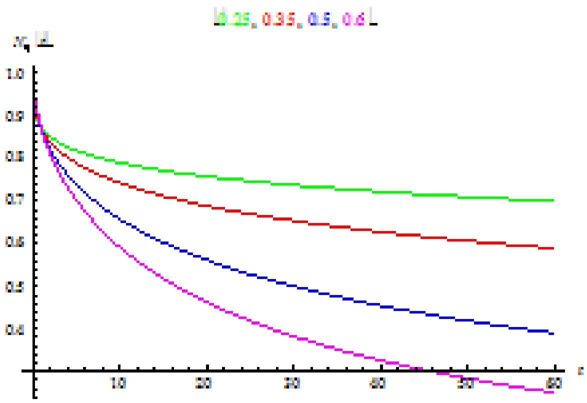

Fig.12 $\mu = 1$, $\gamma = 1$, $q = .5$ and $\nu = \{0.25,\ 0.35,\ 0.5,\ 0.6\}$

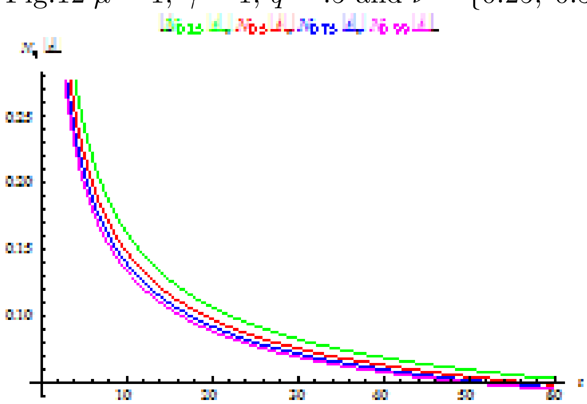

Fig.13 $\mu = .5$, $\gamma = .25$, $\nu = .35$ and $q = \{0.25,\ 0.5,\ 0.75,\ 0.99\}$

Acknowledgment.

The authors would like to thank the anonymous referee for constructive comments that helped us to improve the paper.

References

[1] W.H. Abdi, On certain q-difference equations and q-Laplace transforms, Proc. Natl. Acad. Sci. India, Sect. A, 28 (1962) 1-15.

[2] R.P. Agarwal, Certain fractional q-integrals and q-derivatives, Proc. Camb. Philos. Soc., 66 (1969) 365-370.

[3] G.E. Andrews, q-Series: their development and application in analysis, number theory, combinatorics, physics, and computer algebra, Regional Conference Series in Math. 66, Amer. Math. Soc., 1986.

[4] T. Ernst, The History of q-calculus and a New Method (Licentiate Thesis), U.U.D.M. Report 2000: http://math.uu.se/thomas/Lics.pdf.

[5] H. Exton, q-Hypergeometric Functions and Applications, Ellis Horwood, Chichester, 1983.

[6] R. Floreanini and L. Vinet, On the quantum group and quantum algebra approach to q-special functions, Lett. Math. Phys. 27 (1993) 179–190.

[7] G. Gasper and M. Rahman, 1990, Basic Hypergeometric Series, Cambridge Univ. Press, Cambridge.

[8] W. Hahn, Beitrage Zur Theorie der Heineschen Reihen, die 24 Integrale der hypergeometrischen q-Differenzengleichung, das q-Analogon der Laplace Transformation. Math. Nachr., 2 (1949) 340-379.

[9] H.J. Haubold and A.M. Mathai, The fractional kinetic equation and thermonuclear functions, Astrophys. Space Sci. 327 (2000) 53-63.

[10] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[11] H.T. Koelink, Eight Lectures on Quantum Groups and q-Special Functions, Math. preprint series, Report 96-10, University of Amsterdam, 1996.

[12] V. Kourganoff, Introduction to the Physics of Stellar Interiors (Dordrecht: D. Reidel Publishing Company) , 1973.

[13] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics, in Fractals and Fractional calculus in Continuum Mechanics, A. carpenter and F. Mainardi, Eds., New York: Springer-Verlag, (1997) 291-348.

[14] K. S. Miller and B. Ross, An Introducation to the Fractional Integrals and Derivatives-Theory and Applications, John Willey and Sons, New York, 1993.

[15] I. Podlubny, Fractional Differential Equations, Academic Press, New York, 1999.

[16] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffer function in the kernel, Yokohama Math. J., 19 (1971) 7-15.

[17] L. J. Rogers, Second memoir on the expansion of certain infinite products, Proc. Lond. Math. Soc. (2), 25 (1894), 318-343.

[18] R.K. Saxena, A.M. Mathai and H.J. Haubold, Unified fractional kinetic equation and a fractional diffusion equation, Astrophys. Space Sci., 290 (2004) 299-310.

[19] R.K. Saxena, A.M. Mathai and H.J. Haubold, On fractional kinetic equations, Astrophys. Space Sci., 282 (2002) 281-287.

[20] R.K. Saxena, A.M. Mathai and H.J. Haubold, On generalized fractional kinetic equations, Physica A, 344 (2004) 657-664.

[21] L.J. Slater, Generalized Hypergeometric Functions, Cambridge Univ. Press, Cambridge, 1966.

[22] A. Wiman, ber den Fundamental satz in der Theorie der Funcktionen, $\mathrm{E}_\alpha(\mathrm{x})$, Acta Math., 29 (1905), 191-201.

ЗА ФРАКЦИОНАЛНИТЕ q-КИНЕТИЧКИ РАВЕНКИ

Мирдула Гарг, Лата Чанчлани

Р е з и м е

Во овој труд добиени се решенија на фракционалните q-кинетички равенки кои вклучуваат Риман-Луивилови q-интегрален оператор. Го применуваме методот на q- Лапласова трнсформација и нејзината инверзна за да се добие решение во затворена форма. Решенијата на обичните q- кинетички равенки и на уште две фракционални q-кинетички равенки се добиени како специјален случај на главниот резултат. Исто така нацртавме некои графици на решенијата на фракционалните q- кинетички равенки користејќи го софтверот MATHEMATICA

Department of Mathematics, University of Rajasthan, Jaipur-302004, India
E-mail address: gargmridula@gmail.com

Department of Mathematics, University of Rajasthan, Jaipur-302004, India
E-mail address: chanchlani281106@gmail.com