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A NOTE ON COMPATIBLE BINARY RELATIONS ON VECTOR VALUED HYPERSEMIGROUPS

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Abstract. In this note we present some properties concerning the connection between vector valued hypersemigroups and various kinds of compatible binary relations defined on them, i.e. i-compatible, compatible, strongly i-compatible, strongly compatible, regular and strongly i-regular binary relations.

Binary hyperstructures were introduced by Marty in [8] as a natural extension of classical algebraic structures. Vector valued hyperstructures were introduced in [9] as a generalization of n-ary hyperstructures ([5, 2]) and vector valued structures ([10, 6, 7]). Besides the concepts of vector valued hypergroupoids, hypersemigrops, weak hypersemigroups, etc., regular and strongly regular binary relations on vector valued hypersemigroups were introduced in [9] as well. Following some recent papers of Davvaz and Loreanu-Fotea ([1, 3, 4]), in this short note we introduce the notions of i-compatible, strongly i-compatible, i-regular relations for some $i \in \{0, 1, \ldots, n-1\}$, as well as compatible and strongly compatible relations on vector valued hypersemigroups and prove a few properties concerning these notions. For the sake of completeness, we will repeat the definitions of vector valued hypergroupoid and vector valued hypersemigroup from the paper [9].

Let H be a nonempty set and let n, m be positive integers such that $n \geq m$. Denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of H and by H^n the nth Cartesian product of H.

Definition 1. ([9], Def.1.1.) A mapping $[\,] : H^n \to (\mathcal{P}^*(H))^m$ from the nth Cartesian product of H to the mth Cartesian product of $\mathcal{P}^*(H)$ is called an

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(n, m)-*hyperoperation* on H. If it is not necessary to emphasize the integers n and m, then we will say that $[\ ]$ is a *vector valued hyperoperation* instead of (n, m)-hyperoperation.

Throughout the paper, the elements of H^n, i.e. the sequences $(x_1, \ldots, x_n)$ will be denoted by $x_1x_2\ldots x_n$ or, shortly, x_1^n. The symbol x_i^j will denote the sequence $x_ix_{i+1}\ldots x_j$ of elements of H when $i \leq j$ and the empty symbol when $i > j$.

Definition 2. ([9], Def.1.2.) A sequence of m n-ary hyperoperations $[\ ]_s : H^n \to \mathcal{P}^*(H)$, $s \in \{1, 2, \ldots, m\}$, can be associated to $[\ ]$ by putting

$$[a_1^n]_s = B_s \quad \Leftrightarrow \quad [a_1^n] = (B_1, \ldots, B_m),$$

for all $a_1, \ldots, a_n \in H$. Then, we call $[\ ]_s$ the *sth component hyperoperation* of $[\ ]$ and write $[\ ] = ([\ ]_1, \ldots, [\ ]_m)$. Note that there is a unique (n, m)-hyperoperation on H whose component hyperoperations are $[\ ]_s$.

An (n, m)-hyperoperation $[\ ]$ on H is extended to subsets $A_1, A_2, \ldots, A_n$ of H in a natural way, i.e.

$$[A_1A_2\ldots A_n] = ([A_1A_2\ldots A_n]_1, [A_1A_2\ldots A_n]_2, \ldots, [A_1A_2\ldots A_n]_m),$$

where $[A_1A_2\ldots A_n]_s = \cup\{[a_1^n]_s \mid a_i \in A_i, i = 1, 2, \ldots, n\}$ and $s = 1, 2, \ldots, m$.

Clearly, $C_1^p \subseteq B_1^p$ if and only if $C_i \subseteq B_i$, for $i = 1, \ldots, p$, and, $x_1^p \in C_1^p$ if and only if $x_i \in C_i$ for $i = 1, \ldots, p$.

Definition 3. ([9], Def.1.3.) An algebraic structure $\boldsymbol{H} = (H, [\ ])$, where $[\ ]$ is an (n, m)-ary hyperoperation defined on a nonempty set H, is called an (n, m)-*hypergroupoid* or *vector valued hypergroupoid*. Identifying the set $\{x\}$ with the element x, any (n, m)-groupoid is an (n, m)-hypergroupoid. If $[\ ] = ([\ ]_1, \ldots, [\ ]_m)$, we denote by $(H; [\ ]_1, \ldots, [\ ]_m)$ the *component hypergroupoid* of $\boldsymbol{H}$ and $(H, [\ ]_j)$ is the *jth component n-ary hypergroupoid* of $\boldsymbol{H}$.

Further on we assume that the positive integers n and m are such that $n > m$, i.e. $n = m + k$, for $k \geq 1$.

Definition 4. ([9], Def.1.4.) An (n, m)-hyperoperation is said to be *associative* if

$$[x_1^i[x_{i+1}^{i+n}]x_{i+n+1}^{n+k}] = [x_1^j[x_{j+1}^{j+n}]x_{j+n+1}^{n+k}],$$

holds for all $x_1, \ldots, x_{n+k} \in H$ and for all $i, j \in \{1, 2, \ldots, n\}$.

An (n, m)-hyperoperation is said to be *weakly associative* if

$$[x_1^i[x_{i+1}^{i+n}]x_{i+n+1}^{n+k}]_s \cap [x_1^j[x_{j+1}^{j+n}]x_{j+n+1}^{n+k}]_s \neq \emptyset,$$

holds for all $i, j \in \{1, 2, \dots, n\}$, $x_1, \dots, x_{n+k} \in H$ and every $s \in \{1, 2, \dots, m\}$.

An (n, m)-hypergroupoid with an associative operation (weakly associative operation) is called an (n, m)-*hypersemigroup* (*weak* (n, m)-*hypersemigroup*).

Examples of (n, m)-hypersemigroups and weak (n, m)-hypersemigroups are presented in [9].

Definition 5. ([9], Def.1.10.) Let $(H, [\,])$ and $(H', [\,]')$ be (n, m)-hypergroupoids. A mapping $\varphi : H \to H'$ is:

a) a *strong homomorphism* if and only if $\varphi([a_1^n]_s) = [\varphi(a_1) \dots \varphi(a_n)]'_s$;

b) an *inclusion homomorphism* if and only if $\varphi([a_1^n]_s) \subseteq [\varphi(a_1) \dots \varphi(a_n)]'_s$;

c) a *weak homomorphism* if and only if $\varphi([a_1^n]_s) \cap [\varphi(a_1) \dots \varphi(a_n)]'_s \neq \varnothing$,

for every $s = 1, 2, \dots, n$. The mapping φ that is a bijection and strong homomorphism is called an *isomorphism*, and it is called an *automorphism* if φ is defined on the same (n, m)-hypergroupoid.

Theorem 1. *Let* $\boldsymbol{H}$, $\boldsymbol{H}_1$, $\boldsymbol{H}_2$ *be* (n, m)*-hypersemigroups (weak* (n, m)*-hypersemigroups),* $\varphi_1 : H \to H_1$ *be a surjective strong homomorphism and* $\varphi_2 : H \to H_2$ *be a strong homomorphism, such that* $ker\varphi_1 \subseteq ker\varphi_2$*. Then there exist a unique strong homomorphism* $\theta : H_1 \to H_2$ *such that* $\theta \circ \varphi_1 = \varphi_2$*.*

Proof. Let $a \in H$. Then $\varphi_1(a) = a_1 \in H_1$. Let $\theta : H_1 \to H_2$ be a mapping defined by $\theta(a_1) = \varphi_2(a)$. Let $a_1 = b_1$. Since φ is a surjective mapping it follows that there is $b \in H$ such that $\varphi_1(b) = b_1$. Clearly, $\varphi_1(a) = \varphi_1(b)$, i.e. $(a, b) \in ker\varphi_1 \subseteq ker\varphi_2$. Thus, $\varphi_2(a) = \varphi_2(b)$, i.e. $\theta(a_1) = \theta(b_1)$. Hence, θ is a well defined mapping and

$$(\theta \circ \varphi_1)(a) = \theta(\varphi_1(a)) = \theta(a_1) = \varphi_2(a).$$

The mapping θ is a strong homomorphism. Namely, for every $s \in \{1, 2, \dots, m\}$

$$\theta([a_1^n]_s) = \theta([\varphi_1(a'_1) \dots \varphi_1(a'_n)]_s) = \theta(\varphi_1([a'_1 \dots a'_n]_s)) = \varphi_2([a'_1 \dots a'_n]_s) =$$

$$= [\varphi_2(a'_1) \dots \varphi_2(a'_n)]_s = [(\theta \circ \varphi_1)(a'_1) \dots (\theta \circ \varphi_1)(a'_n)]_s = [\theta(a_1) \dots \theta(a_n)]_s.$$

Suppose that there is a strong homomorphism $\theta_1 : H_1 \to H_2$ such that $\theta_1 \circ \varphi_1 = \varphi_2$. Let $a_1 \in H_1$. Then $\theta_1(a_1) = \theta_1(\varphi_1(a)) = (\theta_1 \circ \varphi_1)(a) = \varphi_2(a) = (\theta \circ \varphi_1)(a) = \theta((\varphi_1)(a)) = \theta(a_1)$, i.e. θ is a unique strong homomorphism. $\square$

Let H be a nonempty set. Denote by $B(H)$ the set of all binary relations on H, by $E(H)$ the set of all equivalence relations on H.

Definition 6. Let $(H, [\ ])$ be an (n, m)-hypersemigroup. A relation $\rho \in B(H)$ is said to be:

a) *i-compatible*, where $i \in \{0, 1, \ldots, n-1\}$, if for any $a, b \in H$ and $s = 1, \ldots, m$

$$(a\rho b \ \wedge \ x \in [x_1^i a x_{i+2}^n]_s) \Rightarrow (\exists y \in [x_1^i b x_{i+2}^n]_s)\ x\rho y.$$

Specially, for $i = 0$ ($i = n-1$) we say that ρ is *right* (*left*) *compatible*.

b) *compatible* if for every $j = 1, 2, \ldots, n$ and $s = 1, \ldots, m$

$$(a_j \rho b_j \ \wedge \ x \in [a_1^n]_s) \Rightarrow (\exists y \in [b_1^n]_s)\ x\rho y.$$

c) *strongly i-compatible* if for any $a, b \in H$

$$a\rho b \Rightarrow x\rho y,$$

for every $x \in [x_1^i a x_{i+1}^n]_s$, $y \in [x_i b x_{i+1}^n]_s$. Specially, for $i = 0$ ($i = n-1$) we say that ρ is *strongly right compatible* (*strongly left compatible*).

d) *strongly compatible* if the following implication holds:

$$(\forall j = 1, \ldots, n) \ \ a_j \rho b_j \ \Rightarrow \ x\rho y,$$

for every $x \in [a_1^n]_s$, $y \in [b_1^n]_s$, $s = 1, \ldots, m$.

If $\rho \in E(H)$ and it is i-compatible, compatible, strongly i-compatible and strongly compatible ($i \in \{0, 1, \ldots, n-1\}$), then ρ is said to be *i-regular*, *regular*, *strongly i-regular*, *strongly regular*, respectively.

Example 1. Let $H = \mathbb{Z}_4$ and $[\ ] : H^3 \to (\mathcal{P}^*(H))^2$ be a $(3, 2)$-hyperoperation defined by:

$$[x_1^3] = \begin{cases} (\{2, 3\}, x_3), & \text{if } \ x_1 = x_2 = x_3 = 0 \\ (\{1, 3\}, x_3), & \text{otherwise.} \end{cases}$$

By a direct verification of each case, one can show: $[[x_1^3]x_4] = (\{1, 3\}, x_4) = [x_1[x_2^4]]$, i.e. $(H, [\ ])$ is a $(3, 2)$-hypersemigroup.

Let $\rho = \{(1, 1), (1, 2), (1, 3), (3, 1)\} \in B(H)$. It can be easily verified that ρ is strongly left compatible (i.e. strongly 2-compatible), since $a\rho b$ implies that $x\rho y$, for every $x \in [x_1^2 a]_s$ and $y \in [x_1^2 b]_s$, $s = 1, 2$. For instance, $(1, 1) \in \rho$ implies that $x\rho y$, for every $x, y \in [x_1^2 1]_1 = \{1, 3\}$ and $x, y \in [x_1^2 1]_2 = 1$. This relation is not strongly 0-compatible or 1-compatible since, for instance, $[2\ 1\ 0]_2 = 0$, $[2\ 2\ 0]_2 = 0$, but $(0, 0) \notin \rho$.

Example 2. Let $H = \{1,2,3,4\}$ and let $[\ ] : H^4 \to (\mathcal{P}^*(H))^2$ be a $(4,2)$-hyperoperation defined by $[x_1^4] = (\{1,2\},\{3,4\})$. Then $(H,[\ ])$ is a $(4,2)$-hypersemigroup. Namely:

$[[x_1^4]x_5^6] = [\{1,2\}\ \{3,4\}\ x_5^6] =$
$= ([13x_5^6]_1 \cup [14x_5^6]_1 \cup [23x_5^6]_1 \cup [24x_5^6]_1, [13x_5^6]_2 \cup [14x_5^6]_2 \cup [23x_5^6]_2 \cup [24x_5^6]_2) =$
$= (\{1,2\},\{3,4\}),$

$[x_1[x_2^5]x_6] = [x_1\ \{1,2\}\ \{3,4\}\ x_6] =$
$= ([x_1 13 x_6]_1 \cup [x_1 14 x_6]_1 \cup [x_1 23 x_6]_1 \cup [x_1 24 x_6]_1,$
$[x_1 13 x_6]_2 \cup [x_1 14 x_6]_2 \cup [x_1 23 x_6]_2 \cup [x_1 24 x_6]_2) = (\{1,2\},\{3,4\}),$

$[x_1^2[x_3^6]] = [x_1^2\ \{1,2\}\ \{3,4\}] =$
$= ([x_1^2 13]_1 \cup [x_1^2 14]_1 \cup [x_1^2 23]_1 \cup [x_1^2 24]_1, [x_1^2 13]_2 \cup [x_1^2 14]_2 \cup [x_1^2 23]_2 \cup [x_1^2 24]_2) =$
$= (\{1,2\},\{3,4\}).$

Let $\rho = \{(1,1),(2,2),(3,3),(4,4),(1,2),(2,1),(3,4),(4,3)\}$ be an equivalence relation on H and let $a_j \rho b_j$, for $j = 1,\ldots,4$. Then $[a_1^4]_1 = [b_1^4]_1 = \{1,2\}$ and $[a_1^4]_2 = [b_1^4]_2 = \{3,4\}$. For every $x \in [a_1^4]_s$ and $y \in [b_1^4]_s$, $s = 1,2$, one obtains that $x\rho y$ holds. Thus, ρ is a strongly regular relation.

Proposition 1. *Let $(H,[\ ])$ be an (n,m)-hypersemigroup. If $\rho \in B(H)$ is reflexive and strongly compatible, then ρ is strongly i-compatible for every $i \in \{0,\ldots,n-1\}$.*

Proof. Let $a\rho b$ for any elements $a,b \in H$ and $x \in [x_1^i a x_{i+2}^n]_s$, $y \in [x_1^i b x_{i+2}^n]_s$ for every $s = 1,\ldots,m$. Since ρ is reflexive, $x_j \rho x_j$, $j \in \{1,\ldots,i,i+2,\ldots,n\}$ and $a\rho b$. The strong compatibility of ρ implies that $a\rho y$. □

Proposition 2. *Let $(H,[\ ])$ be an (n,m)-hypersemigroup and $\rho \in B(H)$ be reflexive and transitive. The relation ρ is strongly compatible if and only if ρ is strongly i-compatible for every $i \in \{0,\ldots,n-1\}$.*

Proof. The direct statement follows from Prop.1. Conversely, let ρ be a reflexive, transitive and strongly i-compatible relation for every $i \in \{0,1,\ldots,n-1\}$. Let $a_j \rho b_j$, $j = 1,\ldots,n$, $x \in [a_1^n]_s$ and $y \in [b_1^n]_s$ for every $s = 1,\ldots,m$. Since:

$$\begin{aligned}
&(a_1\rho b_1 \ \wedge\ x \in [a_1^n]_s \ \wedge\ x_1 \in [b_1 a_2^n]_s) \Rightarrow x\rho x_1,\\
&(a_2\rho b_2 \ \wedge\ x_1 \in [b_1 a_2^n]_s \ \wedge\ x_2 \in [b_1 b_2 a_3^n]_s) \Rightarrow x_1\rho x_2,\\
&\ldots\\
&(a_n\rho b_n \ \wedge\ x_{n-1} \in [b_1^{n-1} a_n]_s \ \wedge\ y \in [b_1^n]_s) \Rightarrow x_{n-1}\rho y,
\end{aligned}$$

and the transitivity of ρ, it follows that $x\rho y$. □

As a consequence of the previous proposition we obtain the following

Corollary 1. *If $\rho \in E(H)$ is a strongly regular relation on (n,m)-hypersemigroup $(H,[\ ])$, then ρ is strongly i-regular relation for every $i \in \{0,\dots,n-1\}$.*

Proposition 3. *Let $(H,[\ ])$ be an (n,m)-hypersemigroup. If $\rho,\theta \in B(H)$ are strongly i-compatible for some $i \in \{0,1,\dots,n-1\}$ (strongly compatible), then $\rho \circ \theta$ is strongly i-compatible (strongly compatible).*

Proof. Let $\rho,\theta \in B(H)$ be strongly i-compatible for some $i \in \{0,\dots,n-1\}$ and $a\ \rho\circ\theta\ b$, $x \in [x_1^i a x_{i+2}^n]_s$, $y \in [x_1^i b x_{i+2}^n]_s$, for every $s = 1,\dots,m$. Since $a\ \rho\circ\theta\ b$, it follows that there exists $c \in H$ such that $a\rho c$ and $c\theta b$. If $z \in [x_1^i c x_{i+1}^n]_s$, then by the strong i-compatibility of ρ it follows that $x\rho z$. One can analogously conclude that $z\theta y$ and thus $x\ \rho\circ\theta\ y$. Strong compatibility can be shown in a similar way. □

Proposition 4. *Let $(H,[\ ])$ be an (n,m)-hypersemigroup. If the relations $\rho_j \in B(H)$, $j \in \{1,\dots,n\}$, are strongly i-compatible for every $i \in \{0,\dots,n-1\}$, then $\bigcup\{\rho_j|\ j=1,\dots,n\}$ is strongly i-compatible.*

Proof. Let $a\ \bigcup_{j=1}^n \rho_j\ b$ and $x \in [x_1^i a x_{i+2}^n]_s$, $y \in [x_1^i b x_{i+2}^n]_s$, for every $s \in \{1,\dots,m\}$. Then, there exists $j \in \{1,\dots,n\}$ such that $a\rho_j b$. Since ρ is a strongly i-compatible relation it follows that $x\rho_j y$ and therefore $x\ \bigcup_{j=1}^n \rho_j\ y$. □

Proposition 5. *Let $(H,[\ ])$ be an (n,m)-hypersemigroup. If the relations $\rho_j \in E(H)$, $j \in \{1,\dots,n\}$, are strongly i-regular for every $i \in \{0,\dots,n-1\}$, then $\bigcap\{\rho_j|\ j=1,\dots,n\}$ is strongly i-regular.*

Proof. Let $a\ \bigcap_{j=1}^n \rho_j\ b$ and $x \in [x_1^i a x_{i+2}^n]_s$, $y \in [x_1^i b x_{i+2}^n]_s$, for every $s \in \{1,\dots,m\}$. Then, for every $j \in \{1,\dots,n\}$, $a\rho_j b$. Since ρ_j are i-regular relations it follows that $x\rho_j y$, for every j. Therefore, $x\ \bigcap_{j=1}^n \rho_j\ y$. □

Theorem 2. *Let **H** and **K** be two (n,m)-hypersemigroups and $\varphi : H \to K$ be a strong homomorphism. Then $\rho = \{(a,b) \in H^2 | \varphi(a) = \varphi(b)\}$ is a regular relation.*

Proof. It is obvious that ρ is an equivalence relation on H. Let $a_j\rho b_j$ for every $j \in \{1,2,\dots,n\}$ and let $x \in [a_1^n]_s$ for $s \in \{1,2,\dots,m\}$. Then $\varphi(a_j) = \varphi(b_j)$. Since $x \in [a_1^n]_s$ it follows that $\varphi(x) \in \varphi([a_1^n]_s)$ and, since φ is a strong homomorphism, we obtain that $\varphi(x) = [\varphi(a_1)\dots\varphi(a_n)]_s =$

$[\varphi(b_1)\dots\varphi(b_n)]_s = \varphi([b_1^n]_s)$. Hence, there exists $y \in [b_1^n]$ such that $\varphi(x) = \varphi(y)$, i.e. $x\rho y$. Therefore, ρ is a regular relation. □

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