A NOTE ON COMPATIBLE BINARY RELATIONS ON VECTOR VALUED HYPERSEMIGROUPS

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Abstract. In this note we present some properties concerning the connection between vector valued hypersemigroups and various kinds of compatible binary relations defined on them, i.e. $i$-compatible, compatible, strongly $i$-compatible, strongly compatible, regular and strongly $i$-regular binary relations.

Binary hyperstructures were introduced by Marty in [8] as a natural extension of classical algebraic structures. Vector valued hyperstructures were introduced in [9] as a generalization of $n$-ary hyperstructures ([5, 2]) and vector valued structures ([10, 6, 7]). Besides the concepts of vector valued hypergroupoids, hypersemigroups, weak hypersemigroups, etc., regular and strongly regular binary relations on vector valued hypersemigroups were introduced in [9] as well. Following some recent papers of Davvaz and Loreanu-Fotea ([1, 3, 4]), in this short note we introduce the notions of $i$-compatible, strongly $i$-compatible, $i$-regular relations for some $i \in \{0, 1, \ldots, n-1\}$, as well as compatible and strongly compatible relations on vector valued hypersemigroups and prove a few properties concerning these notions. For the sake of completeness, we will repeat the definitions of vector valued hypergroupoid and vector valued hypersemigroup from the paper [9].

Let $H$ be a nonempty set and let $n$, $m$ be positive integers such that $n \geq m$. Denote by $\mathcal{P}^*(H)$ the set of all nonempty subsets of $H$ and by $H^n$ the $n$th Cartesian product of $H$.

Definition 1. ([9], Def.1.1.) A mapping $[\ ] : H^n \to (\mathcal{P}^*(H))^m$ from the $n$th Cartesian product of $H$ to the $m$th Cartesian product of $\mathcal{P}^*(H)$ is called an

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A \((n,m)\)-hyperoperation on \(H\). If it is not necessary to emphasize the integers \(n\) and \(m\), then we will say that \([\ ]\) is a \textit{vector valued hyperoperation} instead of \((n,m)\)-hyperoperation.

Throughout the paper, the elements of \(H^n\), i.e. the sequences \((x_1, \ldots, x_n)\) will be denoted by \(x_1 x_2 \ldots x_n\) or, shortly, \(x^n\). The symbol \(x^i_1\) will denote the sequence \(x_i x_{i+1} \ldots x_j\) of elements of \(H\) when \(i \leq j\) and the empty symbol when \(i > j\).

**Definition 2.** ([9], Def.1.2.) A sequence of \(m\) \(n\)-ary hyperoperations \([\ ]_s:\ H^n \rightarrow \mathcal{P}^s(H), s \in \{1, 2, \ldots, m\}\), can be associated to \([\ ]\) by putting

\[a^n_s = B_s \iff [a^n] = (B_1, \ldots, B_m),\]

for all \(a_1, \ldots, a_n \in H\). Then, we call \([\ ]_s\) the \textit{\(s\)th component hyperoperation} of \([\ ]\) and write \([\ ] = ([\ ]_1, \ldots, [\ ]_m)\). Note that there is a unique \((n,m)\)-hyperoperation on \(H\) whose component hyperoperations are \([\ ]_s\).

An \((n,m)\)-hyperoperation \([\ ]\) on \(H\) is extended to subsets \(A_1, A_2, \ldots, A_n\) of \(H\) in a natural way, i.e.

\[A_1 A_2 \ldots A_n = ([A_1 A_2 \ldots A_n]_1, [A_1 A_2 \ldots A_n]_2, \ldots, [A_1 A_2 \ldots A_n]_m),\]

where \([A_1 A_2 \ldots A_n]_s = \bigcup\{[a^n]_s \mid a_i \in A_i, i = 1, 2, \ldots, n\}\) and \(s = 1, 2, \ldots, m\).

Clearly, \(C_i^p \subseteq B_i^p\) if and only if \(C_i \subseteq B_i\), for \(i = 1, \ldots, p\), and, \(x^p_i \in C_i^p\) if and only if \(x_i \in C_i\) for \(i = 1, \ldots, p\).

**Definition 3.** ([9], Def.1.3.) An algebraic structure \(H = (H,[\ ]\))\), where \([\ ]\) is an \((n,m)\)-hyperoperation defined on a nonempty set \(H\), is called an \((n,m)\)-hypergroupoid or \textit{vector valued hypergroupoid}. Identifying the set \(\{x\}\) with the element \(x\), any \((n,m)\)-groupoid is an \((n,m)\)-hypergroupoid. If \([\ ] = ([\ ]_1, \ldots, [\ ]_m)\), we denote by \((H;[\ ]_1, \ldots, [\ ]_m)\) the \textit{component hypergroupoid} of \(H\) and \((H,[\ ]_j)\) is the \(j\)th component \(n\)-ary hypergroupoid of \(H\).

Further on we assume that the positive integers \(n\) and \(m\) are such that \(n > m\), i.e. \(n = m + k\), for \(k \geq 1\).

**Definition 4.** ([9], Def.1.4.) An \((n,m)\)-hyperoperation is said to be \textit{associative} if

\[x^n_i [x_{i+1}^{i+n}]^{i+n} = [x^n_j [x_{j+1}^{j+n}]^{j+n}],\]

holds for all \(x_1, \ldots, x_{n+k} \in H\) and for all \(i, j \in \{1, 2, \ldots, n\}\).

An \((n,m)\)-hyperoperation is said to be \textit{weakly associative} if

\[x^n_i [x_{i+1}^{i+n}]^{i+n} \cap [x^n_j [x_{j+1}^{j+n}]^{j+n}] \neq \emptyset,\]

holds for all \(x_1, \ldots, x_{n+k} \in H\) and for all \(i, j \in \{1, 2, \ldots, n\}\).
holds for all \( i, j \in \{1, 2, \ldots, n\} \), \( x_1, \ldots, x_{n+k} \in H \) and every \( s \in \{1, 2, \ldots, m\} \).

An \((n, m)\)-hypergroupoid with an associative operation (weakly associative operation) is called an \((n, m)\)-hypersemigroup (weak \((n, m)\)-hypersemigroup).

Examples of \((n, m)\)-hypersemigroups and weak \((n, m)\)-hypersemigroups are presented in \([9]\).

**Definition 5.** \([9], \text{Def.1.10}\) Let \((H, [\,])\) and \((H', [\,'])\) be \((n, m)\)-hypergroupoids. A mapping \( \varphi : H \to H' \) is:

a) a strong homomorphism if and only if \( \varphi([a^n]_s) = [\varphi(a_1) \ldots \varphi(a_n)]'_s \);

b) an inclusion homomorphism if and only if \( \varphi([a^n]_s) \subseteq [\varphi(a_1) \ldots \varphi(a_n)]'_s \);

c) a weak homomorphism if and only if \( \varphi([a^n]_s) \cap [\varphi(a_1) \ldots \varphi(a_n)]'_s \neq \emptyset \), for every \( s = 1, 2, \ldots, n \). The mapping \( \varphi \) that is a bijection and strong homomorphism is called an isomorphism, and it is called an automorphism if \( \varphi \) is defined on the same \((n, m)\)-hypergroupoid.

**Theorem 1.** Let \( H, H_1, H_2 \) be \((n, m)\)-hypersemigroups (weak \((n, m)\)-hypersemigroups), \( \varphi_1 : H \to H_1 \) be a surjective strong homomorphism and \( \varphi_2 : H \to H_2 \) be a strong homomorphism, such that \( \ker \varphi_1 \subseteq \ker \varphi_2 \). Then there exist a unique strong homomorphism \( \theta : H_1 \to H_2 \) such that \( \theta \circ \varphi_1 = \varphi_2 \).

**Proof.** Let \( a \in H \). Then \( \varphi_1(a) = a_1 \in H_1 \). Let \( \theta : H_1 \to H_2 \) be a mapping defined by \( \theta(a_1) = \varphi_2(a) \). Let \( a_1 = b_1 \). Since \( \varphi \) is a surjective mapping it follows that there is \( b \in H \) such that \( \varphi_1(b) = b_1 \). Clearly, \( \varphi_1(a) = \varphi_1(b) \), i.e. \( (a, b) \in \ker \varphi_1 \subseteq \ker \varphi_2 \). Thus, \( \varphi_2(a) = \varphi_2(b) \), i.e. \( \theta(a_1) = \theta(b_1) \). Hence, \( \theta \) is a well defined mapping and

\[
(\theta \circ \varphi_1)(a) = \theta(\varphi_1(a)) = \theta(a_1) = \varphi_2(a).
\]

The mapping \( \theta \) is a strong homomorphism. Namely, for every \( s \in \{1, 2, \ldots, m\} \)

\[
\theta([a^n]_s) = \theta([\varphi_1(a'_1) \ldots \varphi_1(a'_n)]_s) = \theta(\varphi_1([a'_1 \ldots a'_n]_s)) = \varphi_2([a'_1 \ldots a'_n]_s) = [\varphi_2(a'_1) \ldots \varphi_2(a'_n)]_s = [((\theta \circ \varphi_1)(a'_1) \ldots (\theta \circ \varphi_1)(a'_n))]_s = [\theta(a_1) \ldots \theta(a_n)]_s.
\]

Suppose that there is a strong homomorphism \( \theta_1 : H_1 \to H_2 \) such that \( \theta_1 \circ \varphi_1 = \varphi_2 \). Let \( a_1 \in H_1 \). Then \( \theta_1(a_1) = \theta_1(\varphi_1(a)) = (\theta_1 \circ \varphi_1)(a) = \varphi_2(a) = (\theta \circ \varphi_1)(a) = \theta((\varphi_1)(a)) = \theta(a_1) \), i.e. \( \theta \) is a unique strong homomorphism. \( \square \)
Let \( H \) be a nonempty set. Denote by \( B(H) \) the set of all binary relations on \( H \), by \( E(H) \) the set of all equivalence relations on \( H \).

**Definition 6.** Let \((H, [\ ]): (n,m)\)-hypersemigroup. A relation \( \rho \in B(H) \) is said to be:

a) \textit{i-compatible}, where \( i \in \{0,1,\ldots, n-1\} \), if for any \( a,b \in H \) and \( s = 1, \ldots, m \)

\[
(apb \land x \in [x^i_1ax^i_{i+2}]_s) \Rightarrow (\exists y \in [x^i_1bx^i_{i+2}]_s) xpy.
\]

Specially, for \( i = 0 \) (\( i = n-1 \)) we say that \( \rho \) is right (left) compatible.

b) \textit{compatible} if for every \( j = 1,2,\ldots, n \) and \( s = 1, \ldots, m \)

\[
(a_jpb_j \land x \in [a^j_3]_s) \Rightarrow (\exists y \in [b^j_3]_s) xpy.
\]

c) \textit{strongly i-compatible} if for any \( a,b \in H \)

\[
apb \Rightarrow xpy,
\]

for every \( x \in [x^i_1ax^i_{i+1}]_s, y \in [x^j_1bx^j_{j+1}]_s \). Specially, for \( i = 0 \) (\( i = n-1 \)) we say that \( \rho \) is \textit{strongly right compatible} (\textit{strongly left compatible}).

d) \textit{strongly compatible} if the following implication holds:

\[
(\forall j = 1, \ldots, n) \ a_jpb_j \Rightarrow xpy,
\]

for every \( x \in [a^j_3]_s, y \in [b^j_3]_s, s = 1, \ldots, m \).

If \( \rho \in E(H) \) and it is \textit{i-compatible}, compatible, strongly \textit{i-compatible} and strongly compatible (\( i \in \{0,1,\ldots, n-1\} \)), then \( \rho \) is said to be \textit{i-regular}, \textit{regular}, \textit{strongly i-regular}, \textit{strongly regular}, respectively.

**Example 1.** Let \( H = \mathbb{Z}_4 \) and \([\ ]): H^3 \rightarrow (P^*(H))^2 \) be a \((3,2)\)-hyperoperation defined by:

\[
[x^3_1] = \begin{cases} 
(\{2,3\},x_3), & \text{if } x_1 = x_2 = x_3 = 0 \\
(\{1,3\},x_3), & \text{otherwise.}
\end{cases}
\]

By a direct verification of each case, one can show: \( [[x^3_1]x_4] = (\{1,3\},x_4) = [x^3_1[x^3_2]], \) i.e. \((H,[\ ])) \) is a \((3,2)\)-hypersemigroup.

Let \( \rho = \{(1,1),(1,2),(1,3),(3,1)\} \in B(H) \). It can be easily verified that \( \rho \) is strongly left compatible (i.e. strongly 2-compatible), since \( apb \) implies that \( xpy \), for every \( x \in [x^1_1a]_s \) and \( y \in [x^2_1b]_s, \) \( s = 1,2 \). For instance, \( (1,1) \in \rho \) implies that \( xpy \), for every \( x,y \in [x^1_11_1] = \{1,3\} \) and \( x,y \in [x^2_11]_2 = 1 \). This relation is not strongly 0-compatible or 1-compatible since, for instance, \( [2 1 0]_2 = 0, [2 2 0]_2 = 0, \) but \( (0,0) \notin \rho \).
Example 2. Let $H = \{1, 2, 3, 4\}$ and let $[\ ] : H^4 \to (\mathcal{P}^*(H))^2$ be a $(4,2)$-hyperoperation defined by $[x^n_i] = (\{1, 2\}, \{3, 4\})$. Then $(H, [\ ])$ is a $(4,2)$-hypersemigroup. Namely:

$$[[x_i^n]x_o^n] = ([1, 2] \{3, 4\} x_o^n) =$$

$$= ([13x_o^n]_1 \cup [14x_o^n]_1 \cup [23x_o^n]_1 \cup [24x_o^n]_1, [13x_o^n]_2 \cup [14x_o^n]_2 \cup [23x_o^n]_2 \cup [24x_o^n]_2) =$$

$$= ([1, 2], \{3, 4\}),$$

$$[x_1^n x_2^n] = [x_1 \{1, 2\} \{3, 4\} x_2^n] =$$

$$= ([x_1^{13} x_6^n]_1 \cup [x_1^{14} x_6^n]_1 \cup [x_1^{23} x_6^n]_1 \cup [x_1^{24} x_6^n]_1,$$

$$[x_1^{13} x_6^n]_2 \cup [x_1^{14} x_6^n]_2 \cup [x_1^{23} x_6^n]_2 \cup [x_1^{24} x_6^n]_2) = ([1, 2], \{3, 4\}),$$

$$[x_1^n x_2^n] = [x_1^n \{1, 2\} \{3, 4\} x_2^n] =$$

$$= ([x_1^{13} x_1^n]_1 \cup [x_1^{14} x_1^n]_1 \cup [x_1^{23} x_1^n]_1 \cup [x_1^{24} x_1^n]_1, [x_1^{13} x_1^n]_2 \cup [x_1^{14} x_1^n]_2 \cup [x_1^{23} x_1^n]_2 \cup [x_1^{24} x_1^n]_2) =$$

$$= ([1, 2], \{3, 4\}).$$

Let $\rho = \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (3, 4), (4, 3)\}$ be an equivalence relation on $H$ and let $a_j \rho b_j$, for $j = 1, \ldots, 4$. Then $[a_1^n]\_1 = [b_1^n]\_1 = \{1, 2\}$ and $[a_1^n]\_2 = [b_1^n]\_2 = \{3, 4\}$. For every $x \in [a_1^n]_s$ and $y \in [b_1^n]_s$, $s = 1, 2$, one obtains that $x \rho y$ holds. Thus, $\rho$ is a strongly regular relation.

**Proposition 1.** Let $(H, [\ ])$ be an $(n, m)$-hypersemigroup. If $\rho \in B(H)$ is reflexive and strongly compatible, then $\rho$ is strongly $i$-compatible for every $i \in \{0, \ldots, n - 1\}$.

Proof. Let $a \rho b$ for any elements $a, b \in H$ and $x \in [x_1^1 a x_i^n, y \in [x_1^1 b x_i^n]_s$, for every $s = 1, \ldots, m$. Since $\rho$ is reflexive, $x_j \rho x_j, j \in \{1, \ldots, i, i + 2, \ldots, n\}$ and $a \rho b$. The strong compatibility of $\rho$ implies that $a \rho y$. ☐

**Proposition 2.** Let $(H, [\ ])$ be an $(n, m)$-hypersemigroup and $\rho \in B(H)$ be reflexive and transitive. The relation $\rho$ is strongly compatible if and only if $\rho$ is strongly $i$-compatible for every $i \in \{0, \ldots, n - 1\}$.

Proof. The direct statement follows from Prop.1. Conversely, let $\rho$ be a reflexive, transitive and strongly $i$-compatible relation for every $i \in \{0, 1, \ldots, n - 1\}$. Let $a_j \rho b_j, j = 1, \ldots, n, x \in [a_1^n]_s$ and $y \in [b_1^n]_s$ for every $s = 1, \ldots, m$. Since:

$$(a_1 \rho b_1 \land x \in [a_1^n]_s \land x_1 \in [b_1^n]_s) \Rightarrow x \rho x_1,$$

$$(a_2 \rho b_2 \land x_1 \in [b_1^n]_s \land x_2 \in [b_1^n]_s) \Rightarrow x_1 \rho x_2,$$

$$\ldots$$

$$(a_n \rho b_n \land x_{n-1} \in [b_1^n]_s \land y \in [b_1^n]_s) \Rightarrow x_{n-1} \rho y,$$
and the transitivity of $\rho$, it follows that $xpy$. \hfill \Box

As a consequence of the previous proposition we obtain the following

**Corollary 1.** If $\rho \in E(H)$ is a strongly regular relation on $(n,m)$-hypersemigroup $(H,\{\})$, then $\rho$ is strongly $i$-regular relation for every $i \in \{0,\ldots, n-1\}$.

**Proposition 3.** Let $(H,\{\})$ be an $(n,m)$-hypersemigroup. If $\rho, \theta \in B(H)$ are strongly $i$-compatible for some $i \in \{0,1,\ldots, n-1\}$ (strongly compatible), then $\rho \circ \theta$ is strongly $i$-compatible (strongly compatible).

*Proof.* Let $\rho, \theta \in B(H)$ be strongly $i$-compatible for some $i \in \{0,\ldots, n-1\}$ and $a \rho \circ \theta \ b$, $x \in [x_i ax_{i+2}]_s$, $y \in [x_i bx_{i+2}]_s$, for every $s \in \{1,\ldots, m\}$. Since $a \rho \circ \theta \ b$, it follows that there exists $c \in H$ such that $apc$ and $c\theta b$. If $z \in [x_i ca_{i+1}]_s$, then by the strong $i$-compatibility of $\rho$ it follows that $xpz$. One can analogously conclude that $\theta y$ and thus $x \rho \circ \theta \ y$. Strong compatibility can be shown in a similar way. \hfill \Box

**Proposition 4.** Let $(H,\{\})$ be an $(n,m)$-hypersemigroup. If the relations $\rho_j \in B(H)$, $j \in \{1,\ldots, n\}$, are strongly $i$-compatible for every $i \in \{0,\ldots, n-1\}$, then $\bigcup \{\rho_j \mid j = 1,\ldots, n\}$ is strongly $i$-compatible.

*Proof.* Let $a \bigcup_{j=1}^n \rho_j \ b$ and $x \in [x_i ax_{i+2}]_s$, $y \in [x_i bx_{i+2}]_s$, for every $s \in \{1,\ldots, m\}$. Then, there exists $j \in \{1,\ldots, n\}$ such that $ap \rho b$. Since $\rho$ is a strongly $i$-compatible relation it follows that $x \rho_j y$ and therefore $x \bigcup_{j=1}^n \rho_j y$. \hfill \Box

**Proposition 5.** Let $(H,\{\})$ be an $(n,m)$-hypersemigroup. If the relations $\rho_j \in E(H)$, $j \in \{1,\ldots, n\}$, are strongly $i$-regular for every $i \in \{0,\ldots, n-1\}$, then $\bigcap \{\rho_j \mid j = 1,\ldots, n\}$ is strongly $i$-regular.

*Proof.* Let $a \bigcap_{j=1}^n \rho_j \ b$ and $x \in [x_i ax_{i+2}]_s$, $y \in [x_i bx_{i+2}]_s$, for every $s \in \{1,\ldots, m\}$. Then, for every $j \in \{1,\ldots, n\}$, $a \rho_j b$. Since $\rho_j$ are $i$-regular relations it follows that $x \rho_j y$, for every $j$. Therefore, $x \bigcap_{j=1}^n \rho_j y$. \hfill \Box

**Theorem 2.** Let $H$ and $K$ be two $(n,m)$-hypersemigroups and $\varphi : H \rightarrow K$ be a strong homomorphism. Then $\rho = \{(a,b) \in H^2 \mid \varphi(a) = \varphi(b)\}$ is a regular relation.

*Proof.* It is obvious that $\rho$ is an equivalence relation on $H$. Let $a_j \rho b_j$ for every $j \in \{1,2,\ldots, n\}$ and let $x \in [a^n_j]_s$ for $s \in \{1,2,\ldots, m\}$. Then $\varphi(a_j) = \varphi(b_j)$. Since $x \in [a^n_j]_s$ it follows that $\varphi(x) \in \varphi([a^n_j]_s)$ and, since $\varphi$ is a strong homomorphism, we obtain that $\varphi(x) = [\varphi(a_1) \ldots \varphi(a_n)]_s$ =
[ϕ(b_1) \ldots \varphi(b_n)]_s = \varphi([b^n_1]_s). \text{ Hence, there exists } y \in [b^n_1]_s \text{ such that } \varphi(x) = \varphi(y), \text{ i.e. } x \rho y. \text{ Therefore, } \rho \text{ is a regular relation.} \square

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