Математички Билтен
42(LXVIII) No. 1
2018(37-44)
Скопје, Македонија

ISSN 0351-336X (print)
ISSN 1857-9914 (online)
UDC: 517.982.4:517.521

SOME RESULTS CONCERNING THE ANALYTIC REPRESENTATION OF CONVOLUTION

VASKO RECKOVSKI, EGZONA ISENI, AND VESNA MANOVA ERAKOVIKJ

Abstract. In this paper we will prove some results concerning the analytic representation of the convolution of some functions and distributions.

1. INTRODUCTION

We use the standard notation from the Schwartz distribution theory.

The boundary value representation has been studied for a long time and from different points of view.

One of the first result is that if $f \in L^1$, then the function

$\hat{f}(z) = \frac{1}{2\pi i} < f(t), \frac{1}{t-z} >$, for $\operatorname{Im} z \neq 0$ is the Cauchy representation of f i.e.

$\lim\limits_{y \to 0^+} < \hat{f}(x+iy) - \hat{f}(x-iy), \varphi(x) >=< f, \varphi >$, for every $\varphi \in D$.

D_{L^p}, $1 \leq p < \infty$ denotes the space of all infinitely differentiable functions φ for which $\varphi^{(\beta)} \in L^p$ for each n-tuple β of nonnegative integers.

$B = D_{L^\infty}$ is the space of all infinitely differentiable functions which are bounded on $\mathbb{R}^n$.

$\dot{B}$ is the subspace of B that consists of all functions $\varphi \in B$ which vanish at infinity together with each of their derivatives.

The topology of D_{L^p} is given in terms of the norms

$\|\varphi\|_{m,\,p} = \left(\int\limits_{\mathbb{R}^n} \left|\varphi^{(\beta)}(x)\right|^p dx \right)^{1/p}, |\beta| \leq m, \quad m = 0, 1, 2, 3, \ldots.$

A sequence of functions (φ_λ) of D_{L^p} converges to a function φ in the topology of D_{L^p}, $1 \leq p < \infty$ as $\lambda \to \lambda_0$ if each $\varphi_\lambda \in D_{L^p}$, $\varphi \in D_{L^p}$, and

$\lim\limits_{\lambda \to \lambda_0} \left\|{\varphi_\lambda}^{(\beta)} - \varphi^{(\beta)}\right\|_{L^p} = \lim\limits_{\lambda \to \lambda_0} \left(\int\limits_{\mathbb{R}^n} \left|{\varphi_\lambda}^{(\beta)}(x) - \varphi^{(\beta)}(x)\right|^p dx \right)^{1/p} = 0$, for every β.

A sequence of functions (φ_λ) converges to the function φ in $\dot{B}$ as $\lambda \to \lambda_0$ if each $\varphi_\lambda \in \dot{B}$, $\varphi \in \dot{B}$, and

2000 Mathematics Subject Classification. Primary 46F20; Secondary 44A15, 46F12.
Key words and phrases. convolution, distribution, analytic representation.

$$\lim_{\lambda\to\lambda_0}\left\|{\varphi_\lambda}^{(\beta)}-\varphi^{(\beta)}\right\|_{L^\infty}=0.$$

${D'}_{L^p}$, $1\leq p<\infty$ is the space of all continuous linear functional on D_{L^q}, where $\frac{1}{p}+\frac{1}{q}=1$. ${D'}_{L^1}$ is the space of all continuous linear functional on $\dot{B}$.

The following theorem gives the structures of ${D'}_{L^p}$.

Structure Theorem. A distribution Λ belongs to ${D'}_{L^p}$, $1\leq p<\infty$ if and only if Λ is a finite sum of distributional derivatives of functions in L^p, i.e. there is an integer $m\geq 0$ depending only on Λ such that

$\Lambda=\sum\limits_{|\beta|\leq m}{f_\beta}^{(\beta)}$, where $f_\beta\in L^p$ for each β, $|\beta|\leq m$.

If $f,g\ \in L^1$ then $\int\limits_{\mathbb{R}}|f(x-y)g(y)|\,dy<\infty$ for almost all x.

The function $h(x)=(g*f)(x)=\int\limits_{\mathbb{R}}f(x-y)g(y)dy$ belongs to $L^1(\mathbb{R})$, $\|h\|_1\leq\|f\|_1\|g\|_1$ and $g*f=f*g$. h is called the convolution of f and g.

If $f\ \in L^1(\mathbb{R})$, $g\in L^p(\mathbb{R})$ for $1\leq p<\infty$ then for almost all $x\in\mathbb{R}^1$, the functions of y, $f(x-y)g(y)$ and $f(y)g(x-y)$ are in $L^1(\mathbb{R})$. For all such x, and $(g*f)(x)=\int\limits_{\mathbb{R}}f(x-y)g(y)dy$ we have $f*g=g*f$ a.e., $f*g\in L^p$ and $\|f*g\|_p\leq\|f\|_1\|g\|_p$.

The proof that $h=f*g$ belongs to L^p for $1\leq p<\infty$ is given in [5].

The following theorem is also known.

Theorem. Let f and g be in L^1 and let $h=g*f=f*g$. Then h has Cauchy representation

$\hat{h}(z)=\frac{1}{2\pi i}\int\frac{h(t)}{t-z}dt=\int\limits_{\mathbb{R}}f(t)\hat{g}(z-t)dt=\int\limits_{\mathbb{R}}g(t)\hat{f}(z-t)dt, z=x+iy$, $\operatorname{Im}z\neq 0$.

2. Main results

We will prove some results concerning the analytic representation of the convolution $h=f*g$ for $f\in L^1$, $g\in L^p$.

Theorem 1. Let $f\in L^1(\mathbb{R})$ and $g\in L^p(\mathbb{R})$, $1\leq p<\infty$ and let $h=f*g$. Then h has the Cauchy representation $\hat{h}(z)=\frac{1}{2\pi i}\int\frac{h(t)}{t-z}dt$,$z=x+iy$, $\operatorname{Im}z\neq 0$.

Proof. For $\varphi\in D$,

$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}[\hat{h}(x+iy)-\hat{h}(x-iy]\varphi(x)dx=$$

$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}\frac{1}{2\pi i}[\int\limits_{\mathbb{R}}(\frac{h(t)}{t-z}-\frac{h(t)}{t-\bar{z}})dt]\varphi(x)dx=$$

$$\lim_{y\to 0^+}\int_{\mathbb{R}}\frac{1}{2\pi i}(\int_{\mathbb{R}}\int_{\mathbb{R}}[\frac{f(u)g(t-u)du}{t-z}-\frac{f(u)g(t-u)du}{t-\bar{z}}]dt)\varphi(x)dx.$$

The above integrals exist by the Hölder inequality, hence applying Fubini's theorem, we may change the order of integration and get that

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy)-\hat{h}(x-iy]\varphi(x)dx=$$
$$\lim_{y\to 0^+}\frac{1}{2\pi i}\int_{\mathbb{R}}(\frac{\varphi(x)}{t-z}-\frac{\varphi(x)}{t-\bar{z}})dx\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}g(t-u)dt=$$
$$\lim_{y\to 0^+}\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)}{|t-z|^2}dx\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}g(t-u)dt.$$

Now by the Lemma 5.4 [1], we get that $\frac{y}{\pi}\int\limits_{\mathbb{R}}\frac{\varphi(x)}{|t-z|^2}dx=\overset{\wedge}{\varphi}(t+iy)$ and that $\int\limits_{\mathbb{R}}f(u)du\int\limits_{\mathbb{R}}g(t-u)\overset{\wedge}{\varphi}(t+iy)dt$ converges to $\int\limits_{\mathbb{R}}f(u)du\int\limits_{\mathbb{R}}g(t-u)\varphi(t)dt$ as $y\to 0^+$.

Finally, with one more use of Fubini's theorem, we get

$$\lim_{y\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+iy)-\hat{h}(x-iy]\varphi(x)dx=$$
$$\int_{\mathbb{R}}f(u)g(t-u)du\int_{\mathbb{R}}\varphi(t)dt=\int_{\mathbb{R}}(f*g)(t)\varphi(t)dt=<f*g,\varphi>.\ \square$$

Theorem 2. Suppose that the sequence $\{f_n\}$ converges to f in L^1 sense and $g\in L^1$. Let $h_n=f_n*g$. Then the sequence $\{h_n\}$ converges to $h=f*g$ in L^1. If $\overset{\wedge}{h_n}(z)$ is analytic representation of every h_n for $n=1,2,3,...$ then the sequence $\left\{\overset{\wedge}{h_n}(z)\right\}$ converges uniformly to $\hat{h}(z)$ on compact subsets of $\mathbb{C}/\mathbb{R}$ and $\hat{h}(z)$ is analytic representation of h.

Proof. Let us consider the difference

$$\left|\int_{\mathbb{R}}h_n(x)dx-\int_{\mathbb{R}}h(x)dx\right|=\left|\int_{\mathbb{R}}(f_n*g)(x)dx-\int_{\mathbb{R}}(f*g)(x)dx\right|$$

$$= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [f_n(y)g(x-y)dy - f(y)g(x-y)dy]dx \right|$$

$$= \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [f_n(y) - f(y)]g(x-y)dydx \right|$$

By Fubini's theorem, in the last integral, we may change the order of integration and get that

$$\left| \int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx \right| \leq \int_{\mathbb{R}} |f_n(y) - f(y)|\, dy \int_{\mathbb{R}} |g(x-y)|\, dx.$$

Since $f_n \to f$ in L^1 and by Lebesgue's dominated convergence theorem we get that $h_n \to h$ in L^1. The facts that the sequence $\left\{ \hat{h}_n(z) \right\}$ converges uniformly to $\hat{h}(z)$ on compact subsets of $\mathbb{C}/\mathbb{R}$ and that $\hat{h}(z)$ is analytic representation of h follow from the Theorem 2.1 in [4] and Theorem 1 in this paper. □

Theorem 3. Suppose that $f \in L^1$ and the sequence $\{g_n\}$, $g_n \in L^p$ for $1 \leq p < \infty$ is such that $g_n \to g$ in L^p. Then the sequence $\{h_n\}$, $h_n = f * g_n$ converges to h in L^p for $1 \leq p < \infty$ and $h = f * g$. If $\hat{h}_n(z)$ is analytic representation of h_n for $n = 1, 2, 3, ...$ respectively, then the sequence $\left\{ \hat{h}_n(z) \right\}$ converges uniformly to $\hat{h}(z)$ on compact subsets of $\mathbb{C}/\mathbb{R}$ and $\hat{h}(z)$ is analytic representation of h.

Proof. Let $h_n = f * g_n$ and $h = f * g$. Then

$$\left| \int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx \right| = \left| \int_{\mathbb{R}} \int_{\mathbb{R}} [f(y)g_n(x-y)dy - f(y)g(x-y)]dydx \right|$$

$$\leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} [f(y)g_n(x-y) - f(y)g(x-y)]dy \right| dx$$

Applying Fubini's theorem in the last integral, we have that

$$\left| \int_{\mathbb{R}} h_n(x)dx - \int_{\mathbb{R}} h(x)dx \right| \leq \int_{\mathbb{R}} |f(y)|\, dy \int_{\mathbb{R}} |g_n(x-y) - g(x-y)|\, dx.$$

Now we may apply the Lebesgue dominated convergence theorem and we get that $h_n \to h$ in L^p sense. The facts that the sequence $\left\{ \hat{h}_n(z) \right\}$

converges uniformly to $\hat{h}(z)$ on compact subsets of $\mathbb{C}/\mathbb{R}$ and that $\hat{h}(z)$ is analytic representation of h follow from the Theorem 2.1 in [4] and Theorem 1 in this paper. □

We denote by $L_Q^p = \left\{ g \mid g \text{ is a measurable function on } \mathbb{R} \text{ and } \frac{g}{Q} \in L^p \right\}$, where Q is a function without real roots.

Theorem 4. Suppose that $f \in L^1(\mathbb{R})$, Q is a function without real roots and g is measurable function on $\mathbb{R}$ that belongs to the space L_Q^p. The convolution of the functions $f \in L^1$ and $\frac{g}{Q} \in L^p$, $h = f * \left(\frac{g}{Q}\right)$ is such that $h \in L^p$, $\|h\|_p \leq \|f\|_1 \left\|\frac{g}{Q}\right\|_p$ and h has Cauchy representation $\hat{h}(z) = \frac{1}{2\pi i} < h, \frac{1}{t-z} >$.

Proof. The fact that $h \in L^p$, $\|h\|_p \leq \|f\|_1 \left\|\frac{g}{Q}\right\|_p$ can be easily proven as in the introduction part.

Let $\varphi \in D$ be arbitrary function. Then we have

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} [\hat{h}(x+i\varepsilon) - \hat{h}(x-i\varepsilon]\varphi(x)dx = \\
&= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} [\frac{1}{2\pi i} \int_{\mathbb{R}} (\frac{h(t)}{t-z} - \frac{h(t)}{t-\bar{z}})dt]\varphi(x)dx = \\
&= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} [\frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dt}{t-z} \int_{\mathbb{R}} f(u)\frac{g(u-t)}{Q(u-t)}du - \\
&\quad - \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{dt}{t-\bar{z}} \int_{\mathbb{R}} f(u)\frac{g(u-t)}{Q(u-t)}du]\varphi(x)dx.
\end{aligned}$$

Since the integrals exist, by Fubini's theorem, we may change the order of integration and get

$$\begin{aligned}
&\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} [\hat{h}(x+i\varepsilon) - \hat{h}(x-i\varepsilon]\varphi(x)dx = \\
&\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}} \varphi(x)dx \frac{y}{\pi} \int_{\mathbb{R}} \frac{dt}{|t-z|^2} \int_{\mathbb{R}} f(u)\frac{g(u-t)}{Q(u-t)}du = \\
&\lim_{\varepsilon \to 0^+} \frac{y}{\pi} \int_{\mathbb{R}} \frac{\varphi(x)dx}{|t-z|^2} \int_{\mathbb{R}} f(u) \int_{\mathbb{R}} \frac{g(u-t)}{Q(u-t)}dudt =
\end{aligned}$$

$$\lim_{\varepsilon\to 0^+}\frac{y}{\pi}\int_{\mathbb{R}}\frac{\varphi(x)dx}{|t-z|^2}\int_{\mathbb{R}}f(u)\frac{g(u-t)}{Q(u-t)}dudt.$$

By the Lebesgue dominated convergence theorem and the Lemma 5.4 in [1], we have that
$\lim\limits_{\varepsilon\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+i\varepsilon)-\hat{h}(x-i\varepsilon)]\varphi(x)dx=\int_{\mathbb{R}}f(u)du\int_{\mathbb{R}}\frac{g(u-t)}{Q(u-t)}\varphi(t)dt..$
One more application of Fubini's theorem gives that

$$\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}}[\hat{h}(x+i\varepsilon)-\hat{h}(x-i\varepsilon)]\varphi(x)dx=$$
$$\int_{\mathbb{R}}f(u)\frac{g(u-t)}{Q(u-t)}du\int_{\mathbb{R}}\varphi(t)dt=$$
$$\int_{\mathbb{R}}(f*\frac{g}{Q})(t)\varphi(t)dt=<f*\frac{g}{Q},\varphi>=<h,\varphi>. \ \square$$

Note. In a similar way, it can be proven another version of this theorem. Namely, if $g\in L^p$ and if f is measurable function such that $f/Q\in L^1$ then the convolution $(f/Q)*g\in L^p$ and, as before, it is proved that has Cauchy representation.

Theorem 5. Suppose that $f\in D'_{L^p}$ and $g\in D'_{L^q}$, where $1\le p<\infty$, $1\le q<\infty$. Then the convolution of the distributions f and g, $h=f*g$, has Cauchy representation.

Proof. For $f\in D'_{L^p}$ and $g\in D'_{L^q}$, we know that $h=f*g\in D'_{L^r}$, $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$.

By the structure theorem, we have that
$f=\sum\limits_{i=1}^{n}f_i{}^{(i)}$ and $g=\sum\limits_{j=1}^{m}g_j{}^{(j)}$, where $f_i{}^{(i)}\in L^p$ and $g_j{}^{(j)}\in L^q$.
Let $\varphi\in D$ be arbitrary function.
Then

$$\lim_{y\to 0^+}\int_{\mathbb{R}}(\hat{h}(x+iy)-\hat{h}(x-iy))\varphi(x)dx=$$
$$\lim_{y\to 0^+}\int_{\mathbb{R}}(\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(t)dt}{t-z}-\frac{1}{2\pi i}\int_{\mathbb{R}}\frac{h(t)dt}{t-\bar{z}})\varphi(x)dx=$$
$$\lim_{y\to 0^+}\int_{\mathbb{R}}\frac{y}{\pi}\int_{\mathbb{R}}\sum_{i=1}^{n}\sum_{j=1}^{m}f_i{}^{(i)}(u)g_j^{(j)}(t-u)dtdu\int_{\mathbb{R}}\frac{dt}{|t-z|^2}\varphi(x)dx=$$

$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\mathbb{R}} f_i{}^{(i)}(u)g_j^{(j)}(t-u)dtdu\frac{y}{\pi}\int\limits_{\mathbb{R}}\frac{dt}{|t-z|^2}\varphi(x)dx.$$

Since the integrals exist, by Fubini's theorem, we may change the order of integration and get

$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}(\hat{h}(x+iy)-\hat{h}(x-iy))\varphi(x)dx =$$
$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}\sum_{i=1}^{n} f_i{}^{(i)}(u)du\int\limits_{\mathbb{R}}\sum_{j=1}^{m} g_j^{(j)}(t-u)dt\frac{y}{\pi}\int\limits_{\mathbb{R}}\frac{\varphi(x)dx}{|t-z|^2}.$$

Using the Lebesgue dominated convergence theorem and the Lemma 5.4, we obtain that

$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}(\hat{h}(x+iy)-\hat{h}(x-iy))\varphi(x)dx =$$
$$\int\limits_{\mathbb{R}}\sum_{i=1}^{n} f_i{}^{(i)}(u)du\int\limits_{\mathbb{R}}\sum_{j=1}^{m} g_j^{(j)}(t-u)\varphi(t)\,dt =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\mathbb{R}} f_i{}^{(i)}(u)du\int\limits_{\mathbb{R}} g_j^{(j)}(t-u)\varphi(t)dt.$$

One more application of Fubini's theorem gives that

$$\lim_{y\to 0^+}\int\limits_{\mathbb{R}}(\hat{h}(x+iy)-\hat{h}(x-iy))\varphi(x)dx =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\mathbb{R}} f_i{}^{(i)}(u)g_j^{(j)}(t-u)du\int\limits_{\mathbb{R}}\varphi(t)dt =$$
$$\sum_{i=1}^{n}\sum_{j=1}^{m}\int\limits_{\mathbb{R}}(f_i{}^{(i)} * g_j^{(j)})(t)\varphi(t)dt =$$
$$\int\limits_{\mathbb{R}} h(t)\varphi(t)dt = < h,\varphi > . \;\square$$

References

[1] H. Bremerman, *Raspredelenija, kompleksnije permenenije i preobrazovanija Furje*, Izdatelstvo "Mir" Moskva, 1968.

[2] R. Carmichael, D. Mitrovic, *Distributions and analytic functions*, New York, 1989.

[3] V. Manova-Erakovic, V. Reckoski, *A note on the analytic representations of convergent sequences in S'*, Filomat 29:6 (2015), University of Nis, Serbia, 2015, p.p. 1419-1424.

[4] V. Manova Erakovik, S. Pilipovic, V. Reckovski, *Analytic representations of sequences in* L^p *spaces,* $1 \leq p < \infty$, Filomat 31:7 (2017), University of Nis, Serbia, 2017, p.p. 1959-1966.

[5] J. Pandey, *The Hilbert transform of Schwartz distributions and applications*, John Wiley & Sons, Inc, 1996.

[6] W. Rudin, *Functional Analysis*, Mc Graw-Hill, Inc., 1970.

Faculty of Tourism and Hospitality,
University St. Kliment Ohridski,
Bitola, Republic of Macedonia
E-mail address: vaskorecko@yahoo.com

University Mother Teresa,
Faculty of Informatics,
ul. 12 Udarna Brigada, br. 2a, kat 7, 1000, Skopje, Republic of Macedonia
E-mail address: egzona.iseni@unt.edu.mk

Ss. Cyril and Methodius University,
Faculty of Mathematics and Natural Sciences,
Arhimedova bb, Gazi baba, 1000, Skopje, Republic of Macedonia
E-mail address: vesname@pmf.ukim.mk