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ABOUT CHARACTERS ON VILENKIN GROUPS

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Abstract. In this paper we proved that Lemma $[7, p.727]$ is not true and that the following Theorem holds: Let G be Vilenkin group. Let $k \in [m_n, m_{n+1})$ be natural number given by $k = \sum_{j=0}^{n} k_j m_j$, $1 \le k_n < p_{n+1}$; $0 \le k_j < p_{+1}$, $j \in \{0, 1, 2, \ldots, n-1\}$ and let $\overline{k}$ be natural number such that $(\forall x \in G)$ $\chi_k(x).\chi_{\overline{k}}(x) = 1$. In this case $\overline{k} = m_{n+1} - k + m_n - 1$. Additionally, we have proved lemma which can be useful independently from our theorem.

1. Introduction and preliminaries

All the research will be performed on Vilenkin group G (i.e. on infinite, totally unconnected, compact Abelian group which satisfies the second axiom of countability). We can introduce the topology on G using zero neighborhood chain

$$G = G_0 \supset G_1 \supset \ldots \supset G_j \supset \ldots \supset \{\theta\}, \bigcap_{j=0}^{\infty} G_j = \{0\} \tag{1}$$

which consists of open subgroups of group G, such that quotient group $G_j \diagup G_{j+1}$ is a cyclic group of prime order $p_{j+1}, \forall j \in N_0$. Hence, for arbitrary $g_j \in G_j \setminus G_{j+1}$ holds

$$G_j = \{[x_j g_j] : 0 \le x_j \le p_{j+1} - 1\}, [x_j g_j] := x_j g_j + G_{j+1} \tag{2}$$

G is said to be bounded iff a sequence

$$(p_j)_{j \in \mathbb{N}} = (p_1, p_2, \ldots)$$

is bounded. For $j \in N$ we denote

$$m_j := p_1 p_2 \ldots p_j \quad (m_0 := 1).$$

Classical example of Vilenkin group is product space

$$\prod_{j=1}^{\infty} G_j,$$

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where $G_j = \{0,1\}$ is a cyclic group of the second order for all $j \in \mathbb{N}$, equipped by discrete topology, with the component adding (note that adding in each component is done by module 2). It's direct generalization is group

$$G = \prod_{k=0}^{\infty} \mathbb{Z}_{n_k},$$

where $\mathbb{Z}_{n_k} := \{0, 1, 2, \ldots, n_k - 1\}, n_k \geq 2$, is a cyclic group of order n_k $(k \in \mathbb{N}_0)$ equipped by discrete topology.

Arbitrary $x \in G$ has the unique representation in the following form

$$x = \sum_{j=0}^{\infty} x_j g_j, \ x_j \in \{0, 1, 2, \ldots, p_{j+1} - 1\} \tag{3}$$

where $g_j \in G_j \diagdown G_{j+1}$ are previously arbitrary chosen and fixed, and for all $j \in \mathbb{N}_0$ holds

$$G_j = \left\{ x \in G : \sum_{s=0}^{\infty} x_s g_s, \ x_s = 0, \text{for } 0 \leq s < j \right\} \tag{4}$$

G is equipped by **Haar measure** μ which is normalized such that for all $j \in \mathbb{N}_0$ holds

$$\mu(G_j) = \frac{1}{m_j}.$$

Clearly,

$$\mu(G) = \mu\left(\bigcup_{j=0}^{\infty} (G_j \diagdown G_{j+1}) \right) = \sum_{j=0}^{\infty} \mu(G_j \diagdown G_{j+1}) = \sum_{j=0}^{\infty} \left(\frac{1}{m_j} - \frac{1}{m_{j+1}} \right) = 1.$$

The class of all continuous functions $f : G \to C$ is denoted by $C(G)$. For each sequence of numbers $(b_j)_{j \in \mathbb{N}}$, with the property $b_n \downarrow 0$, we can introduce metrics in G, invariant in respect to translation, by

$$d(x, y) = \begin{cases} b_j, & x - y \in G_j \diagdown G_{j+1} \\ 0, & x = y. \end{cases}$$

It is a common to take $b_j = \mu(G_j)$ or $b_j = \mu(G_{j+1})$.

Each $n \in \mathbb{N}$ we can uniquely represent in the following form

$$n = \sum_{j=0}^{\infty} n_j m_j, n_j \in \{0, 1, 2, \ldots, p_{j+1} - 1\} \tag{5}$$

where only a finite number of n_j's differ from zero. If $j(n) \in \mathbb{N}_0$ is the smallest of all $s \in \mathbb{N}_0$ with the following property $n_j = 0$ $(\forall j > s)$ $\wedge n_{j(n)} \neq 0$, then

$$n=\sum_{j=0}^{j(n)} n_j m_j, n_j \in \{0,1,2,\ldots,p_{j+1}-1\} \wedge 1 \le n_{j(n)} < p_{j(n)+1} \tag{6}$$

what is equivalent to

$$m_{j(n)} \le n < m_{j(n)+1}.$$

In G there exists a countable collection Γ of **characters** - continuous complex value functions χ, which satisfies the following conditions

$$|\chi(x)| = 1(\forall x \in G), \chi(x+y) = \chi(x)\chi(y)(\forall x, y \in G).$$

The characters form Abelian group in respect to the pointwise product of functions. We topologize $(\Gamma, \cdot)$ by defining neighborhood basis around unit

$$\chi_0 \in \Gamma \ (\chi_0(x) = 1, \forall x \in G)$$

using collection of all sets

$$U(A, \varepsilon) := \{\chi \in \Gamma : |\chi(a) - 1| < \varepsilon, \forall a \in A\},$$

where A goes over collection of all compact subsets in G and ε changes over all positive numbers. It is known ([4, Th.(24.15) and (24.26)]) that $(\Gamma, \cdot)$ is discrete, countable, Abelian group with torsion. Additionally, Vilenkin proved ([8, chapters 1.1 , 1.2, 1.3 and 1.4]) that in Γ exists a chain

$$\Gamma_0 = \{\chi_0\} \subset \Gamma_1 \subset \Gamma_2 \subset \ldots \Gamma_j \subset \ldots \tag{7}$$

consists of subgroups $\Gamma_j = G_j^{\perp}$ of the group Γ, with the following properties:

$$(\forall j \in \mathbb{N}_0) G_j^{\perp} := \{\chi \in \Gamma : \chi(x) = 1, \forall x \in G_j\} = \Gamma(G/G_j), \ \bigcup_{j=0}^{\infty} \Gamma_j = \Gamma \tag{8}$$

$$(\forall j \in \mathbb{N}_0) \Gamma_{j+1} \diagup \Gamma_j \text{ is cyclic group with prime order } p_{j+1} \tag{9}$$

In **ordering** [8], the characters of Γ are ordered as follows: Put $\chi_0(x) = 1(\forall x \in G)$, the constant function, $\Gamma_0 := \chi_0$. Suppose that we have already ordered all the elements of the subgroup

$$\Gamma_j = \{\chi_0, \chi_1, \ldots, \chi_{m_j - 1}\} \tag{10}$$

In $\Gamma_{j+1} \diagdown \Gamma_j$ choose an element χ having minimum order among the elements of $\Gamma_{j+1} \diagdown \Gamma_j$ and denote it by χ_{m_j}. For every k such that $m_{j(n)} \le k < m_{j(n)+1}$, put

$$\chi_k = \chi_{m_{j(k)}}^{k_{j(k)}} \cdot \chi_r,$$

where

$$k = k_{j(k)}.m_{j(k)} + r, \ 1 \le k_{j(k)} < p_{j(k)+1} \wedge 0 \le r < m_{j(k)}.$$

Then all the elements of the subgroup Γ_{j+1} have been ordered and by induction all the elements of Γ. In this kind of ordering, for n given by (5), obviously holds

$$\chi_n = \prod_{j=0}^{\infty} \chi_{m_j}^{n_j}, 0 \le n_j < p_{j+1} \tag{11}$$

and for n given by (6), holds

$$\chi_n = \prod_{j=0}^{j(n)} \chi_{m_j}^{n_j}, 0 \le n_j < p_{j+1} \ (0 \le j < j(n)) \wedge 1 \le n_{j(n)} < p_{j(n)+1} \tag{12}$$

Since $\Gamma_{j+1} \diagup \Gamma_j$ is a cyclic group of prime order p_{j+1}, we have

$$\Gamma_{j+1} \diagdown \Gamma_j = \left\{ \chi_{m_j}^s : 1 \le s \le p_{j+1} - 1 \right\} \tag{13}$$

and

$$\Gamma_{j+1} \diagup \Gamma_j = \left\{ \left[\chi_{m_j}^s \right] : 1 \le s \le p_{j+1} - 1 \right\}, \ \left[\chi_{m_j}^s \right] := \chi_{m_j}^s . \Gamma_j \tag{14}$$

Therefrom, for arbitrary $x_0 \in G_j \diagdown G_{j+1}$ holds $\chi_{m_j}(x_0) \neq 1 \wedge \chi_{m_j}^{p_{j+1}}(x_0) = 1$. This means that $\chi_{m_j}(x_o) = \exp\left(\frac{2\pi i k}{p_{j+1}}\right)$, for some $1 \le k < p_{j+1}$, $i := \sqrt{-1}$. Therefrom, $\left\{ \exp\left(\frac{2\pi i k x_j}{p_{j+1}}\right) : 1 \le x_j \le p_{j+1} - 1 \right\}$ is a set of all primitive $p_{j+1} - th$ roots of 1 and using (2) follows $\{x_j x_0 : 1 \le x_j \le p_{j+1} - 1\} = G_j \diagdown G_{j+1}$. Hence, χ_{m_j} takes, on $G_j \diagdown G_{j+1}$, all values from the set $\left\{ \exp\left(\frac{2\pi i s}{p_{j+1}}\right) : 1 \le s \le p_{j+1} - 1 \right\}$ and only these values. Because of that

$$\exists g_j \in G_j \diagdown G_{j+1} \text{ such that } \chi_{m_j}(g_j) = \exp\left(\frac{2\pi i}{p_{j+1}}\right) \tag{15}$$

Hereafter we will always by g_j denote element from (15). Now it is clear

$$\sum_{x_j=1}^{p_{j+1}-1} \chi_{m_j}^{x_j}(g_j) = -1 \text{ ,or } \sum_{x_j=0}^{p_{j+1}-1} \chi_{m_j}^{x_j}(g_j) = 0 \tag{16}$$

Without loss of generality we can assume that g_j, appearing in (3), are exactly those g_j with the property (15), and furthermore we will assume that this is fulfilled. Now, arbitrary $x \in G$, given by (3), we can identify with the sequence

$$x = (x_j)_{j \in \mathbb{N}_0}, \ x_j \in \{0, 1, 2, \ldots, p_{j+1} - 1\} \tag{17}$$

of the corresponding coefficients. Addition of elements $x = (x_j)_{j\in\mathbb{N}_0}$ and $y = (y_j)_{j\in\mathbb{N}_0}$ in G we denote by $+$ and define it by component addition. Component addition for arbitrary coordinate $j \in \mathbb{N}_0$ is addition $\bmod p_{j+1}$.

For arbitrary $n \in \mathbb{N}$ given by (5) and arbitrary $x \in g$ given by (3) we have

$$\chi_n(x) = \prod_{j=0}^{\infty} \chi_{m_j}^{n_j}(x),\ 0 \le n_j < p_{j+1} \tag{18}$$

Analogously, for arbitrary $n \in \mathbb{N}$ given by (6) and arbitrary $x \in g$ given by (3) we have

$$\chi_n(x) = \prod_{j=0}^{j(n)} \chi_{m_j}^{n_j}(x), 0 \le n_j < p_{j+1}\ (0 \le j < j(n)) \wedge 1 \le n_{j(n)} < p_{j(n)+1} \tag{19}$$

If we for convenience, for arbitrary $j \in \mathbb{N}_0$ put

$$r_j := \chi_{m_j} \tag{20}$$

then, (11) becomes

$$\chi_n = \prod_{j=0}^{\infty} r_j^{n_j},\ 0 \le n_j < p_{j+1} \tag{11*}$$

(12) becomes

$$\chi_n = \prod_{j=0}^{j(n)} r_j^{n_j}, 0 \le n_j < p_{j+1}\ (0 \le j < j(n)) \wedge 1 \le n_{j(n)} < p_{j(n)+1} \tag{12*}$$

(15) becomes

$$\exists g_j \in G_j \diagdown G_{j+1} \text{ such that } r_j(g_j) = \exp\left(\frac{2\pi i}{p_{j+1}}\right) \tag{15*}$$

(16) becomes

$$\sum_{x_j=1}^{p_{j+1}-1} r_j^{x_j}(g_j) = -1, \text{ or } \sum_{x_j=0}^{p_{j+1}-1} r_j^{x_j}(g_j) = 0 \tag{16*}$$

(18) becomes

$$\chi_n(x) = \prod_{j=0}^{\infty} r_j^{n_j}(x),\ 0 \le n_j < p_{j+1}, \tag{18*}$$

(19) becomes

$$\chi_n(x) = \prod_{j=0}^{j(n)} r_j^{n_j}(x), 0 \le n_j < p_{j+1}\ (0 \le j < j(n)) \wedge 1 \le n_{j(n)} < p_{j(n)+1} \tag{19*}$$

Let us remark that, according to the way how we introduce elements ordering in Γ, follows that, without loss of generality we can assume that

$$(\forall n \in \mathbb{N}_0) r_n^{p_{n+1}} = r_{n-1},\ r_{-1} := \chi_0 \tag{21}$$

For $k \in \mathbb{N}_0$ let $\overline{k} \in \mathbb{N}_0$ be defined by

$$\chi_k(x).\chi_{\overline{k}}(x) = 1,\ \text{for all } x \in G \text{ i.e. } \chi_k \cdot \chi_{\overline{k}} = \chi_0 \tag{22}$$

Remark 1. *The importance of the element* $\overline{\chi_k} = \chi_{\overline{k}}$ *is noticed, for example in* [2, *p.4, relation(1.5) and Theorm1.*]. *Probably because of that the author of the paper* [7, *p.727,* ***Lemma 7.***] *for arbitrary chosen*

$$k \in \mathbb{N},\ k \in [m_j, m_{j+1}) \wedge k = \sum_{i=0}^{j} k_i m_i,\ 1 \le k_j < p_{j+1};\ 0 \le k_i < p_{i+1}, 0 \le i \le j-1$$

tried to explicitly calculate corresponding $\overline{k}$. *But, obtained result*

$$\overline{k} = m_{j+1} - k + \sum_{i=0,\ k_i \ne 0}^{j-1} m_{i+1} \tag{23}$$

is not true.

2. Results

Proposition 1. *Lemma 7* [7, *p*.727] *is not true.*

Proof. Let us construct Vilenkin group for which **Lemma 7.**[7, *p*.727] does not hold.

Let G Vilenkin group for which: $p_1 = 2 \wedge\ p_2 = 3 \wedge p_j$ is a arbitrary prime number for all $j \ge 3$. Then:

a) $m_1 = 2 \wedge m_2 = 6 \wedge m_3 = 6p_3 \wedge ...$
b) $0 \le n_0 < 2 \wedge 0 \le n_1 < 3 \wedge 0 \le n_2 < p_3 \wedge ...$
c) $[m_0, m_1) = [1, 2) \wedge [m_1, m_2) = [2, 6) \wedge [m_2, m_3) = [6, 6p_3) \wedge ...$

d) $\Gamma_0 = \{\chi_0\} \wedge \Gamma_1 = \{\chi_0, \chi_1\} \wedge \Gamma_2 = \{\chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5\} \wedge \Gamma_3 = \{\chi_j : 0 \le j < m_3\} \wedge$ ...

e) $\Gamma_1 \diagdown \Gamma_0 = \{\chi_1\} \wedge \Gamma_2 \diagdown \Gamma_1 = \{\chi_2, \chi_3, \chi_4, \chi_5\} \wedge \Gamma_3 \diagdown \Gamma_2 = \{\chi_6, \chi_7, ..., \chi_{m_3-1}\} \wedge$ ...

f) $\forall j \in \mathbb{N}_0 \Gamma_{j+1} \diagup \Gamma_j$ is a cuclic group with prime order p_{j+1}. Then:
   i) $\Gamma_1 \diagup \Gamma_0$ is a cuclic group with prime order $p_1 = 2$, then $\chi_1^2 = \chi_0$,
   ii) $\Gamma_2 \diagup \Gamma_1$ is a cuclic group with prime order $p_2 = 3$, then $\chi_2^3 = \chi_3^3 = \chi_4^3 = \chi_5^3 = \chi_1$.

g) By (6) and (12), in the case when $k = 5 = 1.1 + 2.2 = k_0 m_0 + k_1 m_1$, we would have $\chi_k = \chi_5 = \chi_1 . \chi_2^2$ and according to (23) $\overline{k} = m_2 - k + m_1 = 6 - 5 + 2 = 3$. Therefore $\chi_k . \chi_{\overline{k}} = \chi_5 . \chi_3 = \chi_1 . \chi_2^2 . \chi_1 . \chi_2 = \chi_1^2 . \chi_2^3 = \chi_0 . \chi_1 = \chi_1 \neq \chi_0$

□

Theorem 1. *Let G be Vilenkin group. Let $k \in [m_n, m_{n+1})$ be natural number given by*

$$k = \sum_{j=0}^{n} k_j m_j, 1 \le k_n < p_{n+1};\ 0 \le k_j < p_{+1},\ j \in \{0, 1, 2, \dots, n-1\}$$

and let $\overline{k}$ be natural number such that

$$(\forall x \in G) \chi_k(x) . \chi_{\overline{k}}(x) = 1.$$

Then

$$\overline{k} = m_{n+1} - k + m_n - 1 \tag{24}$$

Before we start with the proof let us formulate and prove following lemma (some parts of mentioned lemma we will use in the proof of the theorem).

Lemma 1. *a)* $(n \in \mathbb{N}_0) \wedge (\forall s \in \mathbb{N}_0 : 0 \le s \le n)\ r_n^{\frac{m_{n+1}}{m_s}} \in \Gamma_s \diagdown \Gamma_{s-1}$ $(\Gamma_{-1} := \theta)$.

b) $(n \in \mathbb{N}_0) \wedge (\forall s \in \mathbb{N}_0)\ r_n(x_s g_s) = \begin{cases} 1, & s > n; \\ \exp\left(2\pi i \frac{x_s . m_s}{m_{n+1}}\right), & s \le n. \end{cases}$

c) $(n \in \mathbb{N}_0) \wedge (\forall x \in G)\ r_n(x) = \exp\left(2\pi i \dfrac{\sum_{j=0}^{n} x_j m_j}{m_{n+1}}\right)$.

d) Let $k \in [m_n, m_{n+1})$ be natural number given by

$$k = \sum_{j=0}^{n} k_j m_j, 1 \le k_n < p_{n+1}; 0 \le k_j < p_{+1}, j \in \{0, 1, 2, ..., n-1\}.$$

Then, for each $x \in G$ (given by (3)*) holds*

$$\chi_k(x) = \exp\left(2\pi i \sum_{j=0}^{n} \left(\frac{k_j}{m_{j+1}} \cdot \sum_{s=0}^{j} x_s m_s\right)\right).$$

e) For $n \in N$ given by (5) *and $x \in G$ given by* (3) *holds*

$$\chi_n(x) = \exp\left(2\pi i \sum_{j=0}^{\infty}\left(\frac{k_j}{m_{j+1}} \cdot \sum_{s=0}^{j} x_s m_s\right)\right).$$

Proof. (Lemma 1).

a) By the facts that $r_n \in \Gamma_{n+1} \diagdown \Gamma_n$ and $\Gamma_{n+1} \diagup \Gamma_n$ is a cyclic group of prime order p_{n+1} follows $r_n^{p_{n+1}} \in \Gamma_n \diagup \Gamma_{n-1}$. Therefrom and using the fact that $\Gamma_n \diagup \Gamma_{n-1}$ is a cyclic group of prime order p_n follows $r_n^{p_{n+1} \cdot p_n} \in \Gamma_{n-1} \diagup \Gamma_{n-2}$. By continuing reasoning describe above we conclude that for each $0 \leq s \leq n$ holds $r_n^{p_{n+1} \cdot p_n \cdot \ldots \cdot p_{s+1}} \in \Gamma_s \diagup \Gamma_{s-1}$. This proves assertion *a*) since obviously holds

$$(\forall s \in \{0, 1, 2, \ldots, n\}) p_{n+1} \cdot p_n \ldots p_{s+1} = \frac{m_{n+1}}{m_s}.$$

Let us notice, that from (21) follows

$$r_n^{p_{n+1} \cdot p_n \cdots p_{s+1}} = r_s^{p_{s+1}} = r_{s-1}.$$

b) If $s > n$, then $r_n \in \Gamma_s$ and therefrom $r_n(x_s g_s) = 1$ since $x_s g_s \in \Gamma_s \diagdown \Gamma_{s+1}$. Let us arbitrarily choose $s \in \{0, 1, 2, \ldots, n\}$. Then $x_s g_s \in \Gamma_s \diagdown \Gamma_{s+1}$, for all $1 \leq x_s < p_{s+1}$. Therefrom and using *a*) follows $r_n^{\frac{m_{n+1}}{m_s}}(x_s g_s) = 1$, i.e. $r_s^{p_{s+1}}(x_s g_s) = 1$, or $r_s(x_s g_s) = \exp(2\pi i \frac{t_s}{p_{s+1}})$ for some $1 \leq t_s < p_{s+1}$. Therefrom and bay (15*) follows $\exp(2\pi i \frac{x_s}{p_{s+1}}) = \exp(2\pi i \frac{t_s}{p_{s+1}})$. Thereby an by $1 \leq x_s < p_{s+1}$ and $1 \leq t_s < p_{s+1}$ follows $t_s = x_s$. By arbitrariness of $s \in \{0, 1, 2, \ldots, n\}$, follows

$$(\forall s \in \{0, 1, 2, \ldots, n\}) t_s = x_s.$$

This completes the proof of assertion *b*).

c)

$$\begin{aligned} (3) \wedge b) \Rightarrow r_n(x) &= r_n\left(\sum_{j=0}^{\infty} x_j g_j\right) = \prod_{j=0}^{n} r_n(x_j g_j) \\ &= \prod_{j=0}^{n} \exp(2\pi i \frac{x_j m_j}{m_{n+1}}) = \exp\left(2\pi i \frac{\sum_{j=0}^{n} x_j m_j}{m_{n+1}}\right). \end{aligned}$$

d) By (19*) and according to Lemma 1 (part *c*)) we have

$$\begin{aligned}\chi_k(x) &= \prod_{j=0}^{n} r_j^{k_j}(x) = \prod_{j=0}^{n} \exp\left(2\pi i \frac{k_j \cdot \sum\limits_{s=0}^{j} x_s m_s}{m_{j+1}}\right) \\ &= \exp\left(2\pi i \sum_{j=0}^{n}\left(\frac{k_j}{m_{j+1}} \cdot \sum_{s=0}^{j} x_s m_s\right)\right).\end{aligned}$$

e) By (5) $\wedge$ (18*) $\wedge$ c) we have

$$\begin{aligned}\chi_n(x) &= \prod_{j=0}^{\infty} r_j^{k_j}(x) = \prod_{j=0}^{\infty} \exp\left(2\pi i \frac{k_j \cdot \sum\limits_{s=0}^{j} x_s m_s}{m_{j+1}}\right) \\ &= \exp\left(2\pi i \sum_{j=0}^{\infty}\left(\frac{k_j}{m_{j+1}} \cdot \sum_{s=0}^{j} x_s m_s\right)\right).\end{aligned}$$

□

Proof. (Theorem 1). Let G be given Vilenkin group. Let $k \in [m_n, m_{n+1})$ be natural number given by

$$k = \sum_{j=0}^{n} k_j m_j,\ 1 \leq k_n < p_{n+1};\ 0 \leq k_j < p_{j+1},\ j \in \{0, 1, 2, \ldots, n-1\}$$

and let $\overline{k}$ be natural number such that (22) holds. By (19*) we have

$$(\forall x \in G)\chi_k(x) = \prod_{j=0}^{n} r_j^{k_j}(x) \tag{25}$$

For arbitrary $x \in G_n \diagdown G_{n+1}$ holds $\chi_k(x) \neq 1$ (since, $r_n(x) \neq 1 \wedge 1 \leq k_n < p_{n+1}$). Thereby and from (22) follows $(\forall x \in G_n \diagdown G_{n+1})\, \chi_{\overline{k}}(x) \neq 1$. This means that $\chi_{\overline{k}} \notin \Gamma_n$. So, $\overline{k} \geq m_n$. On the other hand, for all $x \in G_{n+1}$ holds $\chi_k(x) = 1$ (since, $r_j \in \Gamma_{n+1} = G_{n+1}^{\perp}$, for al $0 \leq j \leq n$). Therefrom and by (22) follows $(\forall x \in G_{n+1})\, \chi_{\overline{k}}(x) = 1$. This means that $\chi_{\overline{k}} \in \Gamma_{n+1}$, and hence $\overline{k} < m_{n+1}$. So,

$$\overline{k} \in [m_n, m_{n+1}).$$

According to (6) we conclude that $\overline{k}$ has unique representation in the form

$$\overline{k}=\sum_{j=0}^{n} b_j m_j, 1\leq b_n<p_{n+1};\ 0\leq b_j<p_{j+1},\ j\in\{0,1,2,\dots,n-1\} \tag{26}$$

Applying our Lemma 1 and equality (22), let us determine coefficients b_j for all $j\in\{0,1,2,\dots,n\}$. By Lemma 1 (part d)) and by (25), for each $x\in G$ (given by (3)) holds

$$\left.\begin{array}{c}\chi_k(x)=\exp\left(2\pi i\sum_{j=0}^{n}\left(\frac{k_j}{m_{j+1}}\cdot\sum_{s=0}^{j}x_s m_s\right)\right)\wedge\\ \chi_{\overline{k}}(x)=\exp\left(2\pi i\sum_{j=0}^{n}\left(\frac{b_j}{m_{j+1}}\cdot\sum_{s=0}^{j}x_s m_s\right)\right)\end{array}\right\} \tag{27}$$

By (27) follows, for each $x\in G$ (given by (3)) holds true

$$\chi_k(x)\cdot\chi_{\overline{k}}(x)=\exp\left(2\pi i\sum_{j=0}^{n}\left(\frac{k_j+b_j}{m_{j+1}}\cdot\sum_{s=0}^{j}x_s m_s\right)\right) \tag{28}$$

By (28) it is clear that (22) is satisvied if and only if

$$\sum_{j=0}^{n}\left(\frac{k_j+b_j}{m_{j+1}}\cdot\sum_{s=0}^{j}x_s m_s\right)\in\mathbb{N}_0 \tag{29}$$

Coefficient b_n can be determined as follows. Let $x\in G$ such that

$$x=(0,0,\dots,0,g_n,0,\dots).$$

For such x we have $x_n=1\wedge x_s=0$ for all $s\neq n$, so in that case from (28) we obtain

$$\frac{k_n+b_n}{m_{n+1}}\cdot m_n\in\mathbb{N}_0,\ \text{which is equivalent to}\ \frac{k_n+b_n}{p_{n+1}}\in\mathbb{N}_0.$$

If we here take in account that $1\leq k_n< p_{n+1}\wedge 1\leq b_n< p_{n+1}$, we can conclude

that it has to be

$$k_n+b_n=p_{n+1},\ \text{i. e.}\ b_n=p_{n+1}-k_n \tag{30}$$

Coefficient b_{n-1} is determined as follows . Let us choice $x\in G$ such that

$$x = (0, 0, \dots, 0, g_{n-1}, 0, \dots).$$

For such x we have $x_{n-1} = 1 \wedge x_s = 0$ for all $s \neq n-1$, so in that case (29) implies

$$\frac{k_{n-1} + b_{n-1}}{m_n} \cdot m_{n-1} + \frac{k_n + b_n}{m_{n+1}} \cdot m_{n-1} \in \mathbb{N}_0,$$

which is according to (30), equivalent to

$$\frac{k_{n-1} + b_{n-1}}{p_n} + \frac{1}{p_n} \in \mathbb{N}_0, \text{ i.e. } \frac{k_{n-1} + b_{n-1} + 1}{p_n} \in \mathbb{N}_0.$$

If we take account that $0 \leq k_{n-1} < p_n \wedge 1 \leq b_{n-1} < p_n$, we conclude that is has to be

$$k_{n-1} + b_{n-1} + 1 = p_n, \text{ i.e. } b_{n-1} = p_n - k_{n-1} - 1 \tag{31}$$

Coefficient b_{n-2} is determine analogusly. So, let us choice $x \in G$ such that

$$x = (0, 0, \dots, 0, g_{n-2}, 0, \dots).$$

For such x we have $x_{n-2} = 1 \wedge x_s = 0$ for all $s \neq n-2$, so in that case (29) implies

$$\frac{k_{n-2} + b_{n-2}}{m_{n-1}} \cdot m_{n-2} + \frac{k_{n-1} + b_{n-1}}{m_n} \cdot m_{n-2} + \frac{k_n + b_n}{m_{n+1}} \cdot m_{n-2} \in \mathbb{N}_0,$$

which is acording to (30) $\wedge$ (31), equivalent to

$$\frac{k_{n-2} + b_{n-2}}{p_{n-1}} + \frac{p_{n-1}}{p_{n-1}p_n} + \frac{p_{n+1}}{p_{n-1}p_n p_{n+1}} \in \mathbb{N}_0, \text{ i. e. } \frac{k_{n-2} + b_{n-2} + 1}{p_{n-1}} \in \mathbb{N}_0.$$

If we here take in account fact

$$0 \leq k_{n-2} < p_{n-1} \wedge 1 \leq b_{n-2} < p_{n-1},$$

we can conclude that is has to be

$$k_{n-2} + b_{n-2} + 1 = p_{n-1}, \text{ i.e. } b_{n-2} = p_{n-1} - k_{n-2} - 1. \tag{32}$$

By continuing this process, as many times as we need, we finally conclude

$$b_n = p_{n+1} - k_n \wedge (\forall j \, \{0, 1, 2, \dots, n-1\}) b_j = p_{j+1} - k_j - 1. \tag{33}$$

Substituing (33) in (26) we obtain

$$\overline{k} = (p_{n+1} - k_n)m_n + \sum_{j=0}^{n-1} (p_{j+1} - k_j - 1)m_j = m_{n+1} - k + m_n - 1.$$

Now it is easy to check, that for $\overline{k}$ determined in a such way, really,

$$\chi_k \cdot \chi_{\overline{k}} = \chi_0$$

holds true. Namelly,

$$\chi_k \cdot \chi_{\overline{k}} = \left(\prod_{j=0}^{n} r_j^{k_j} \right) \cdot \left(\prod_{j=0}^{n} r_j^{b_j} \right) = r_n^{p_{n+1}} \cdot \left(\prod_{j=0}^{n-1} r_j^{p_{j+1}-1} \right) = \chi_0$$

(according to (23) and (21). □

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