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A NEW PROOF FOR THE EULER THEOREM IN THE COMPLEX NUMBERS THEORY

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Abstract. In this paper, a new proof for the Euler equation ($\exp(ix) = \cos x + i \sin x$) has been presented. At first, a new and general formula has been proved from which the Euler equation has been derived.

1. Introduction

Euler equation in the theory of the complex numbers is usually proved by expansion of $\sin(x)$, $\cos(x)$ and $\exp(x)$ into power series. A general proof of this equation based on direct mathematical analysis does not exist. In this paper, at first a new formula has been proved from which the Euler equation has been derived as a special result.

2. Analysis

Let f be an analytic function with the following characteristics

$$f(z) = u(x,y) + iv(x,y), f(z) \neq \pm iz_0, z_0 = a + ib \neq 0, z = x + iy, i = \sqrt{-1} \quad (2.1)$$

Since f is an analytic function [1]

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (2)$$

U and V are defined as follows

$$U(\phi,\varphi) = \frac{\phi}{\phi^2+\varphi^2}, V(\phi,\varphi) = -\frac{\varphi}{\phi^2+\varphi^2}, \phi = \phi(x,y), \varphi = \varphi(x,y) \Rightarrow$$

$$\begin{aligned} \frac{\partial U}{\partial x} &= \frac{(\varphi^2-\phi^2)\frac{\partial \phi}{\partial x} - 2\phi\varphi\frac{\partial \varphi}{\partial x}}{(\phi^2+\varphi^2)^2}, \frac{\partial U}{\partial y} = \frac{(\varphi^2-\phi^2)\frac{\partial \phi}{\partial y} - 2\phi\varphi\frac{\partial \varphi}{\partial y}}{(\phi^2+\varphi^2)^2} \\ \frac{\partial V}{\partial x} &= -\frac{(\phi^2-\varphi^2)\frac{\partial \varphi}{\partial x} - 2\phi\varphi\frac{\partial \phi}{\partial x}}{(\phi^2+\varphi^2)^2}, \frac{\partial V}{\partial y} = -\frac{(\phi^2-\varphi^2)\frac{\partial \varphi}{\partial y} - 2\phi\varphi\frac{\partial \phi}{\partial y}}{(\phi^2+\varphi^2)^2} \end{aligned} \quad (3)$$

Let define g as

$$g(z) = \frac{1}{f^2(z)+z_0^2} = \frac{1}{\phi_1 + i\varphi_1} = U(\phi_1,\varphi_1) + iV(\phi_1,\varphi_1),$$

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$\phi_1 = u^2 - v^2 + a^2 - b^2$, $\varphi_1 = 2(uv + ab)$
Using Eq. 2

$$\begin{aligned}\frac{\partial\phi_1}{\partial x} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}, \frac{\partial\phi_1}{\partial y} = -2u\frac{\partial v}{\partial x} - 2v\frac{\partial u}{\partial x},\\ \frac{\partial\varphi_1}{\partial x} = 2u\frac{\partial v}{\partial x} + 2v\frac{\partial u}{\partial x}, \frac{\partial\varphi_1}{\partial y} = 2u\frac{\partial u}{\partial x} - 2v\frac{\partial v}{\partial x}\end{aligned} \tag{4}$$

From Eqs. 3 and 4

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = 2\frac{(\varphi_1^2 u - \phi_1^2 u - 2\phi_1\varphi_1 v)\frac{\partial u}{\partial x} + (-\varphi_1^2 v + \phi_1^2 v - 2\phi_1\varphi_1 u)\frac{\partial v}{\partial x}}{(\phi_1^2 + \varphi_1^2)^2}$$

$$\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = 2\frac{-(\varphi_1^2 u - \phi_1^2 u - 2\phi_1\varphi_1 v)\frac{\partial v}{\partial x} + (-\varphi_1^2 v + \phi_1^2 v - 2\phi_1\varphi_1 u)\frac{\partial u}{\partial x}}{(\phi_1^2 + \varphi_1^2)^2}$$

Therefore, g is an analytic function. Let define h as follows

$$h(z) = \frac{1}{f(z) + iz_0} = \frac{1}{\phi_2 + i\varphi_2} = U(\phi_2, \varphi_2) + iV(\phi_2, \varphi_2),\ \phi_2 = u - b,\ \varphi_2 = v + a$$

Using Eq. 2

$$\frac{\partial\phi_2}{\partial x} = \frac{\partial u}{\partial x}, \frac{\partial\phi_2}{\partial y} = -\frac{\partial v}{\partial x}, \frac{\partial\varphi_2}{\partial x} = \frac{\partial v}{\partial x}, \frac{\partial\varphi_2}{\partial y} = \frac{\partial u}{\partial x} \tag{5}$$

From Eqs. 3 and 5

$$\frac{\partial U}{\partial x} = \frac{\partial V}{\partial y} = \frac{(\varphi_2^2 - \phi_2^2)\frac{\partial u}{\partial x} - 2\phi_2\varphi_2\frac{\partial v}{\partial x}}{(\phi_2^2 + \varphi_2^2)^2}, \frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x} = \frac{-(\varphi_2^2 - \phi_2^2)\frac{\partial v}{\partial x} - 2\phi_2\varphi_2\frac{\partial u}{\partial x}}{(\phi_2^2 + \varphi_2^2)^2}$$

Therefore, h is an analytic function. Let define s as

$$s(z) = \frac{1}{f(z) - iz_0} = \frac{1}{\phi_3 + i\varphi_3} = U(\phi_3, \varphi_3) + iV(\phi_3, \varphi_3),\ \phi_3 = u + b, \varphi_3 = v - a$$

Like the procedure was used for $h(z)$, it can be shown similarly that $s(z)$ is also an analytic function. Since $f(z)$ is an analytic function, for any continuous curve

C from z_0 to z [1]

$$\int_C f(z)dz = \int_{Z_0}^{Z} f(z)dz = F(z) - F(z_0) = F(z) + c_0, F'(z) = f(z)$$

$$g(z)f'(z) = h(z)s(z)f'(z) = \frac{1}{2iz_0}(s(z) - h(z))f'(z)$$

$$\Rightarrow \int_C \frac{f'(z)dz}{f^2(z) + z_0^2} = \frac{1}{2iz_0} \int_C \left(\frac{f'(z)}{f(z) - iz_0} - \frac{f'(z)}{f(z) + iz_0} \right) dz + c_0$$

$$\Rightarrow \frac{1}{z_0} \tan^{-1} \frac{f(z)}{z_0} + c_0 = \frac{1}{2iz_0} \ln \frac{f(z) - iz_0}{f(z) + iz_0}$$

$$\Rightarrow \frac{f(z) - iz_0}{f(z) + iz_0} = e^{2i \tan^{-1} \frac{f(z)}{z_0} + 2ic_0 z_0}, \; for\, f(z) = 0$$

$$\Rightarrow -1 = e^{2ic_0 z_0}$$

$$\Rightarrow \frac{f(z) - iz_0}{f(z) + iz_0} = -e^{2i \tan^{-1} \frac{f(z)}{z_0}}, \; f(z) \neq \pm iz_0 \qquad (6)$$

The function $f(z)$ can be defined as

$$f(z) = z_0 \tan(p(z)/2) \Rightarrow \frac{f(z) - iz_0}{f(z) + iz_0} = -\cos p(z) - i \sin p(z)$$

and

$$-e^{2i \tan^{-1} \frac{f(z)}{z_0}} = -e^{ip(z)} \Rightarrow e^{ip(z)} = \cos p(z) + i \sin p(z)$$

References

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