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APPLICATION OF SCHWARZ FUNCTION ON AN INVERSE BOUNDARY VALUE PROBLEM

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Abstract

The Schwarz function for the curve is a unique analytic function $S(z)$, that in every point z of the curve its value is $\overline{z}$. The complex equation $\overline{z} = g(z)$ will be called K-contour, if it describes closed or non-closed contour or a set of isolated points, when $g(z)$ is an analytic function. In the case of inverse boundary problem when a simple smooth contour is given, the problem of finding unknown K-contour and unknown poly-analytic function under the given boundary conditions is solved.

1. Introduction

Schwarz functions appear in the case of transforming the equation of a simple, smooth, closed (or non-closed) real function L: $F(x,y) = 0$ into complex form. By using conjugate complex variables

$$z = x + iy, \qquad \overline{z} = x - iy,$$
$$x = \frac{1}{2}(z + \overline{z}), \quad y = \frac{1}{2i}(z - \overline{z}), \tag{1.1}$$

function F transforms into

$$f(x,y) = f\Big(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\Big) = G(z, \overline{z}) = 0,$$

which, under some conditions can be expressed via $\overline{z}$ as

$$\overline{z} = S(z), \tag{1.2}$$

where $S(z)$ is analytic function of complex variables in some domain Ω. For example, the complex form of some real curves is as follows:

a) A line crossing points z_1 and z_2

$$\overline{z} = S(z) = \left(\frac{\overline{z}_1 - \overline{z}_2}{z_1 - z_2}\right)(z - z_2) + \overline{z}_2\,; \tag{1.3}$$

b) A circle with radius r and a center in the point z_0

$$\overline{z} = S(z) = \frac{r^2}{z - z_0} + \overline{z}_0\,; \tag{1.4}$$

c) An ellipse $(x^2/y^2) + (y^2/b^2) = 1, \quad (a > b)$

$$\overline{z} = S(z) = \frac{a^2 + b^2}{a^2 - b^2}\,z + \frac{2ab}{b^2 - a^2}\sqrt{z^2 + b^2 - a^2}\,; \tag{1.5}$$

d) A hyperbola $x^2 - y^2 = a^2$

$$\overline{z} = S(z) = \sqrt{2a^2 - z^2}\,; \tag{1.6}$$

e) An equation of conic section $\mathrm{ax}^2 + 2bxy + cy^2 = 1$

$$z^2(a - c - 2bi) + 2z\overline{z}(a + c) + \overline{z}^2(a - c + 2bi) = 4\,. \tag{1.7}$$

All equations (1.3) – (1.7) are self-conjugated, which means that by its conjugation the same equation is obtained. The functions $S(z)$ on the right side in mentioned equations are Schwarz functions for given curves.

Let L be a simple, smooth and closed contour. An analytic function is unique defined if the value in every point on the contour is defined. The Schwarz function for the curve L can be defined as a unique analytic function $S(z)$, that in every point z of the curve L is $\overline{z}$.

Let $g(z)$ be analytic function such that complex equation

$$\overline{z} = g(z) \tag{1.8}$$

describes closed or non-closed contour or a set of isolated points. Further, the set of points in the complex plane defined by (1.8) will be called K-contour (see [1]). It is clear that an arbitrary analytic function $g(z)$, in general case, cannot be a Schwartz function, because the condition of self-conjugation must be satisfied. But the condition of self-conjugation is not necessity, because there are self-conjugated functions that do not represent real curves. For example, $G(z, \overline{z}) = z\overline{z} + 1 = 0$ is a self-conjugated function, but it is not a real curve.

Let f be conformal transformation that real segment $a \le t \le b$ transforms into a simple smooth curve L. The condition of sufficiency and necessity for the analytic function $S(z)$ to be a Schwarz function is to be in the form

$$S = \overline{f} f^{-1} \tag{1.9}$$

(see [2]).

The Schwarz function has many applications, and one of these is in the theory of boundary value problems. In this paper an inverse boundary value problem for poly-analytic functions is solved.

2. Inverse Problem for Poly-analytic Functions

Definition 2.1. Let $g(z)$ be a given analytic function in some domain Ω, and let $w = w(z, \bar{z})$ be a continuous complex function that permits a convergent power series expansion in terms of z and $\bar{z}$ in Ω. Then, the compound function $w(z, g(z))$ is an analytic function and let it be denoted by $\alpha_{g(z)}w$. The operator $\alpha_{g(z)}w$ maps the set of continuous complex functions $w = w(z, \bar{z})$ to the set of analytic functions, and the geometric meaning of this operator is as follows: if $\bar{z} = g(z)$ is an equation of a closed contour Γ, then functions $w = w(z, \bar{z})$ and $\alpha_{g(z)}w$ have the same value on Γ.

It is known that the boundary value problems in Theory of analytic functions reduce on finding analytic function or reveal some of its properties on the base of the functional relations defined on a given contour. On the other hand, in the case of inverse boundary problems, it is necessary to find the contour L of the region on the base of the boundary conditions that are given on L. In such an inverse problem, analytic (or non-analytic) function or the domain can be search, where one condition more than in the case of ordinary boundary problem must be set. This extra boundary condition is used for determining unknown domain's contour.

Basically, the problems of inverse boundary problems for analytic functions or for simple, smooth and closed contours are considered. In some papers, different generalizations of the inverse boundary problems and application in mechanics and technology are considered.

In this paper the inverse boundary value problem for poly-analytic function and for K-contour is solved.

Prooblem P. Let Γ: $\bar{z} = h(z)$ be a simple, smooth contour. The problem is to find unknown K-contour L: $\bar{z} = g(z)$ and unknown poly-analytic function

$$w(z, \bar{z}) = f_0(z) + \bar{z} f_1(z) + \bar{z}^2 f_2(z) + \cdots + \bar{z}^n f_n(z) \tag{2.1}$$

under the given boundary conditions

$$\alpha_{g(z)}w = \varphi_0(z),\ \alpha_{g(z)}Dw = \varphi_1(z), \ldots, \alpha_{g(z)}D^n w = \varphi_n(z) \tag{2.2}$$

$$\alpha_{h(z)}w = \psi(z)\,, \tag{2.3}$$

where $\varphi_0(z), \varphi_1(z), \ldots, \varphi_n(z)$, $\psi(z)$ are given analytic functions and $Dw = (u'_x - v'_y) + i(u'_y + v'_x)$ is the Kolosov differential operator.

It is obtain from (2.1)

$$\begin{aligned} Dw &= 2f_1(z) + 2 \cdot 2\bar{z} f_2(z) + \cdots + 2n\bar{z}^{n-1} f_n(z) \\ D^2 w &= 2^2 \cdot 2 f_2(z) + \cdots + 2^2 n(n-1)\bar{z}^{n-2} f_n(z) \\ &\vdots \\ D^n w &= 2^n n!\, f_n(z) \end{aligned}$$

and

$$\left.\begin{aligned}\alpha_{g(z)}w &= f_0(z)+g(z)f_1(z)+\cdots+g^n(z)f_n(z)=\varphi_0(z)\\ \alpha_{g(z)}Dw &= \qquad 2f_1(z)+\cdots+2ng^{n-1}(z)f_n(z)=\varphi_1(z)\\ &\vdots\\ \alpha_{g(z)}D^nw &= \qquad\qquad 2^n n!\, f_n(z)=\varphi_n(z)\end{aligned}\right\} \tag{2.4}$$

$$\alpha_{h(z)}w = f_0(z)+h(z)f_1(z)+\cdots+h^n(z)f_n(z)=\psi(z)\,. \tag{2.5}$$

The condition (2.4) is a linear system of $(n+1)$ equations with $(n+1)$ unknown functions $f_0(z), f_1(z), \ldots, f_n(z)$. The determinant of the system

$$\begin{vmatrix} 1 & g(z) & g^2(z) & \cdots & g^n(z)\\ 0 & 2 & 2\cdot 2g(z) & \cdots & 2ng^{n-1}(z)\\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 2^n n! \end{vmatrix} = 1\cdot 2\cdot 2^2\cdot 2!\ldots 2^n n! \neq 0\,,$$

so, the system has an unique solution.

From the last condition of (2.4) it obtains

$$f_n(z)=\varphi_n(z)/(2^n n!)=a_0(z)\,,$$

which, by substituting in previous condition in (2.4) gives

$$f_{n-1}(z)=a_1(z)+g(z)b_1(z)\,.$$

The same procedure yields

$$f_{n-2}(z)=a_2(z)+g(z)b_2(z)+g^2(z)c_2(z)\,,$$

and finally

$$f_0(z)=a_n(z)+g(z)b_n(z)+g^2(z)c_n(z)+\cdots+g^{n-1}(z)p_n(z)+g^n(z)q_n(z)\,,$$

where the coefficients $a_k, b_k, \ldots, p_k, q_k$ are expressed via $\varphi_0(z), \varphi_1, \ldots, \varphi_n(z)$. By substitution of $f_0(z), f_1(z), \ldots, f_n(z)$ in condition (2.5), after some calculations the algebraic equation

$$A_n(z)g^n(z)+A_{n-1}(z)g^{n-1}(z)+\cdots+A_1(z)g(z)+A_0(z)=0 \tag{2.6}$$

is obtained where the coefficients $A_k(z)$, $(k=0,1,\ldots,n)$ are expressed via the functions $a_k, b_k, \ldots, p_k, q_k$.

If the equation (2.6) has n different solutions $g_1(z)\neq g_2(z)\neq\cdots\neq g_n(z)$, then there exists n different K-contours denoted as L_1: $\overline{z}=g_1(z), \ldots, L_n$: $\overline{z}=g_n(z)$ and n different solutions of the given inverse boundary problem (2.1)–(2.2)–(2.3). As it can be seen, the problem of existence and the number of solutions of the problem P is reduced to discussion of solvable of the equation (2.6). So, for every solution $g_k(z)$ of (2.6) corresponds a system of values $f_0(z), f_1(z), \ldots, f_n(z)$ in corresponding poly-analytic function (2.1).

Example Let Γ: $\overline{z} = 4/z$ be a given circle. The problem is to find unknown curve L: $\overline{z} = g(z)$ and unknown bi-analytic function

$$w(z,\overline{z}) = f_0(z) + \overline{z} f_1(z) \tag{2.7}$$

that satisfies the following conditions

$$\begin{aligned} \alpha_{g(z)} w &= \exp(z) + z \\ \alpha_{g(z)} Dw &= 2z^2 \\ \alpha_{4/z} w &= \exp(z) + 4z\,. \end{aligned} \tag{2.8}$$

Substituting $\alpha_{g(z0} w = f_0(z) + g(z) f_1(z), Dw = 2f_1(z)$ and $\alpha_{g(z)} Dw = 2f_1(z)$ in boundary conditions (2.8) gives

$$\begin{aligned} f_0(z) + g(z) f_1(z) &= \exp(z) + z \\ 2f_1(z) &= 2z^2 \\ f_0(z) + (4/z) f_1(z) &= \exp(z) + 4z, \end{aligned}$$

and after some calculations it is found that

$$f_0(z) = \exp(z), \quad f_1(z) = z^2, \quad g(z) = 1/z.$$

So, the solution of this example is bi-analytic function $w(z,\overline{z}) = \exp(z) + \overline{z}z^2$, and the searched contour L is an unit circle $\overline{z} = g(z) = 1/z$. For this circle, $g(z) = 1/z$ is Schwarz function. In this example $z = f(t) = \exp(it)$, so the condition (1.9) is satisfied.

3. Conclusion

In this paper some basic information of application of Schwarz function $\overline{z} = S(z)$ in the Theory of boundary value problems is presented. The given example is the simplest one, treating the linear case of algebraic equation (2.6). The following issues of application of Schwarz function on boundary value problems seem to be interesting for further investigation: a) The procedure developed here permits to determine K-contour $\overline{z} = g(z)$ that may be reduced on the set of isolated points. Thus, the basic problem is more general: to examine if for some real simple smooth curve, derived function $g(z)$ is Schwarz one. In the given Example the answer is positive and corresponding curve is unit circle; b) Consider other types of direct and inverse boundary value problems being formulated for other classes of non-analytic functions (p-analytic, meta-analytic etc.)

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ПРИМЕНА НА ШВАРЦОВАТА ФУНКЦИЈА НА ЕДЕН ГРАНИЧЕН ПРОБЛЕМ

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Р е з и м е

Шварцова функција за една крива е еднозначна аналитичка функција $S(z)$, која во секоја точка z од кривата има вредност $\bar{z}$. Комплексната равенка $\bar{z} = g(z)$ се нарекува K-контура ако таа е затворена или отворена контура или пак множество од изолирани точки, кога $g(z)$ е аналитичка функција. Во случај на инверзен граничен проблем кога е дадена проста глатка контура, решен е проблемот за наоѓање на непозната K-контура и непозната полианалитичка функција при дадени гранични услови.

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