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A CRITERION FOR POLYNOMIAL DECOMPOSITION

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Abstract

Let $B = B(x)$ be a complex polynomial for which $\deg B(x) = m \cdot n$, $m \geq 1$, $n \geq 2$, $m, n \in N$. In this work we state a criterion for the following proposition to hold:

$$\begin{cases} \text{there exist complex polynomials } y = y(x),\ \deg y(x) = m \\ \text{and } u = u(x),\ \deg u(x) = n, \text{ such that} \\ B(x) = u\big(y(x)\big). \end{cases}$$

In addition, as an auxilary result we obtain a theorem that completely solves the problem of the polynomial solutions of the algebraic equation

$$B(x) = c_0 + c_1 \cdot y + \cdots + c_{n-1} \cdot y^{n-1} + c_n \cdot y^n,$$

giving also an algorithm for finding them.

1. All the polynomials, considered in this work (except the ones from remark in section 2), will be complex (i.e. elements of $C[x]$) and that will not be mentioned below.

Let $B = B(x)$ be a given complex polynomial (i.e. an element of $C[x]$) for the degree of which it holds:

$$\deg B = m \cdot n, \qquad m \geq 1, \quad n \geq 2, \quad m, n \in N \tag{1.1}$$

Keywords: polynomial decomposition, polynomial solution of algebraic equation, polynomial part of *a* n–th root of polynomial.

The main goal of this work is to establish a criterion for decomposition of $B(x)$ into two polynomials of degrees m and n i.e. a criterion for the following condition to hold:

$$\begin{cases} \text{there exist complex polynomials } y = y(x), \ \deg y(x) = m \\ \text{and } u = u(x), \deg u(x) = n, \ \text{such that} \\ B(x) = u\big(y(x)\big) \end{cases} \qquad \ldots(D)$$

For the purpose of this paper, we use the following equivalent form of the condition (D):

$$\begin{cases} \text{there exist complex constants } \ b_0, b_1, \ldots, b_n (b_n \neq 0) \\ \text{and a complex polynomial } \ y = y(x) \text{ such that:} \\ B(x) = b_0 + b_1 \cdot y + b_2 \cdot y^2 + \cdots + b_{n-1} \cdot y^{n-1} + b_n \cdot y^n \end{cases} \qquad (1.2)$$

By $\left[\sqrt[n]{B(x)}\right]$ we shall denote the polynomial part of the expansion of $\sqrt[n]{B(x)}$ in decreasing powers of x. More precisely, we have the following definition:

Definition: If

$$B(x) = \beta_0 \cdot x^{m\cdot n} + \beta_1 \cdot x^{m\cdot n-1} + \cdots + \beta_{m\cdot n-1} \cdot x + \beta_{m\cdot n}\,, \quad \beta_0 \neq 0,$$

$$\beta_i \in C (i = 0, 1, \ldots, m \cdot n), \quad m \geq 1, \ n \geq 2, \text{ then :}$$

$$\left[\sqrt[n]{B(x)}\right] = p \cdot p \Big\{ \sqrt[n]{\beta_0} \cdot x^m \cdot \Big[1 + \binom{1/n}{1} \cdot \Big(\frac{\beta_1}{\beta_0 \cdot x} + \cdots + \frac{\beta_{m\cdot n}}{\beta_0 \cdot x^{m\cdot n}}\Big) + $$
$$+ \binom{1/n}{2} \cdot \Big(\frac{\beta_1}{\beta_0 \cdot x} + \cdots + \frac{\beta_{m\cdot n}}{\beta_0 \cdot x^{m\cdot n}}\Big)^2 + \cdots \Big]\Big\},$$

where $p \cdot p$ stands for the polynomial part, and $\sqrt[n]{\beta_0}$ is some fixed value of a root of degree n of β_0. Therefore, $\left[\sqrt[n]{B(x)}\right]$ is a plynomial of degree m determined up to a factor which is an n–th root of 1. The polynomials S and Q are specified as:

$$S = \left[\sqrt[n]{B(x)}\right], \qquad B = S^n + Q\,. \qquad (1.3)$$

In several works, starting with [6] (where $n = 2$ is considered), the polynomials S and Q, determined as (1.3), are used to describe some polynomials solutions of algebraic differential equations of the Riccati type.

If $A = A(x)$ is a polynomial, then we shall denote by $(A)_i$ its mean on the interval $[0, 1]$:

$$(A)_i = \int_0^1 A(t) \cdot dt\,. \tag{1.4}$$

Using the notations introduced, we can formulate the main results of this work.

Theorem 1. Let $B = B(x)$ be a polynomial whose degree satisfies (1.1). Then (D) holds iff there exists a complex constant c such that:

$$\Gamma_{[0,1]}\left(Q-c,\, S,\, S^2,\, \ldots,\, S^{n-2}\right) =$$

$$= \begin{vmatrix} (Q\overline{Q})_i - c(\overline{Q})_i - \overline{c}(Q)_i + c\,\overline{c} & (Q\overline{S})_i - c(\overline{S})_i & \ldots & (Q\overline{S}^{n-2})_i - c(\overline{S}^{n-2})_i \\ (S\overline{Q})_i - \overline{c}(S)_i & (S \cdot \overline{S})_i & \ldots & (S \cdot \overline{S}^{n-2})_i \\ \ldots & \ldots & \ldots & \ldots \\ (S^{n-2}\overline{Q})_i - \overline{c}(S^{n-2})_i & (S^{n-2} \cdot \overline{S})_i & \ldots & (S^{n-2} \cdot \overline{S}^{n-2})_i \end{vmatrix} = 0 \tag{1.5}$$

where the polynomials S and Q are specified in (1.3). It can be seen (e.g. by looking at the expansion of this determinant by the first column) that the left–hand side of (1.5) is of the type:

$$\alpha \cdot c \cdot \overline{c} + \beta \cdot c + \gamma \cdot \overline{c} + \delta\,.$$

Therefore (D) is reduced to the condition that an equation of the type:

$$\alpha \cdot x \cdot \overline{x} + \beta \cdot + \gamma \cdot \overline{x} + \delta = 0 \tag{1.6}$$

with known complex constant α, β, γ, δ, have roots in C and easily verifiable (establishing the conditions for (1.6) to have roots in C is the simplest exercise).

Theorem 1 for $n = 2$ can be stated in the following simple form.

Corollary 1. Let $B = B(x)$ be a polynomial of degree m, $m \geq 1$. Then

$$\begin{cases} \text{there exist complex constant } b_0,\, b_1,\, b_2 (b_2 \neq 0), \\ \text{and a polynomial } y = y(x), \text{ such that} \\ B = b_0 + b_1 \cdot y + b_2 \cdot y^2\,, \end{cases}$$

iff

$$B(x) - \left[\sqrt[n]{B(x)}\right]^2 = \text{const.}$$

We shall prove Theorem **1** in next section using another result, which is significant in its own right.

2. In this section we formulate necessary and sufficient conditions for the algebraic equation

$$B(x) = c_0 + c_1 \cdot y + c_2 \cdot y^2 + \cdots + c_{n-1} \cdot y^{n-1} + c_n \cdot y^n , \tag{2.1}$$

where $n \geq 2$, $c_0, c_1, \ldots, c_{n-1}, c_n$ are complex constants ($c_n \neq 0$) and $B = B(x)$ is a polynomial, to have polynomial solutions. We also determine the explicit form of the polynomial solutions of (2.1).

If the degree of the polynomial $B = B(x)$ is not a multiple of n, then, clearly, equation (2.1) has no polynomial solutions (exsept the trivial case when $B(x)$ is a constant). Therefore we shall restrict our attention to the case $\deg B = m \cdot n$, $m \geq 1$. We shall also suppose that $c_n = 1$, which does not diminish the generality of the problem considered.

Theorem **2**. Let $B = B(x)$ be a polynomial, $n \geq 2$ be a natural number and $a_0, a_1, \ldots, a_{n-2}$ be a complex constants. If $\deg B = m \cdot n$, $m \geq 1$, then the equation:

$$B(x) = a_0 + a_1 \cdot v + a_2 \cdot v^2 + \cdots + a_{n-2} \cdot v^{n-2} + v^n \tag{2.2}$$

has polynomial solutions v iff there exist a number $t \in \{1, 2, \ldots, n\}$ suh that:

$$Q = a_0 + a_1(\omega_t S) + a_2(\omega_t S)^2 + \cdots + a_{n-2}(\omega_t S)^{n-2} , \tag{2.3}$$

where S and Q are given by (1.3), and $\omega_1, \omega_2, \ldots, \omega_n$ are all the n^{th} roots of 1; also, if for some $t = t_0 (a \leq t_0 \leq n)$ (2.3) holds, then the polynomial $v = \omega_{t_0} \cdot S$ is an solution of (2.2) and this equation cannot have polynommial solutions other than $v = \omega_t \cdot S$, $t = 1, 2, \ldots, n$.

Corollary **2**. Let $B = B(x)$ be a polynomial, $c_n = 1$, $c_0, c_1, \ldots, c_{n-1}$ complex constants. If $\deg B = m \cdot n$, $m \geq 1$, $n \geq 2$, then equation (2.1) has a polynomial solution iff there exists a number $t \in \{1, 2, \ldots, n\}$ sush that (2.3) holds, where the constants $a_0, a_1, \ldots, a_{n-2}$ are specified as:

$$a_{n-k} = \sum_{i=0}^{k} \binom{n-i}{k-i} \cdot c_{n-i} \cdot \alpha^{k-i} , \quad c_n = 1 ,$$

$$\alpha = -\frac{c_{n-1}}{n} , \quad k = 2, 3, \ldots, n ; \tag{2.4}$$

if (2.3) holds for some $t \in \{1, 2, \ldots, n\}$, then the polynomial

$$y = \omega_t \cdot S - \frac{c_{n-1}}{n} \tag{2.5}$$

is a solution of (2.1) and this equation cannot have polynomial solutions other than the functions (2.5) for $t = 1, 2, \ldots, n$.

So Corollary **2** completely solves the problem of polynomial solutions of the algebraic equation (2.1), including the algorithm for finding them. Let us note that the polynomial solutions of equation (2.1) (for $c_n = 1$), when they exist, do not depend on the constants $c_0, c_1, \ldots, c_{n-2}$, but only on c_{n-1} and $B(x)$.

Example. For the equation

$$B \equiv x^6 + (2+3i)\cdot x^4 + (-6+4i)\cdot x^2 - 2 - 4i = \\ = y^3 + 2\cdot y^2 - 3\cdot y \tag{2.6}$$

we have:

$$S = \left[\sqrt[3]{B}\right] = x^2 + \frac{1}{3}\cdot(2+3i), \quad Q = B - S^3 = -\frac{13}{3}\cdot x^2 - \frac{8}{27} - \frac{13}{3}i,$$

$$c_0 = 0, \quad c_1 = -3, \quad c_2 = 2, \quad a_0 = \frac{70}{27}, \quad a_1 - \frac{13}{3},$$

so the condition (2.3) is ($n = 3$):

$$-\frac{13}{3}\cdot x^2 - \frac{8}{27} - \frac{13}{3}i = \frac{70}{27} - \frac{13}{3}\cdot\omega_t\cdot\left(x^2 + \frac{2+3i}{3}\right), \quad (\omega_t^3 = 1,\ t = 1,2,3),$$

which is an identity for $\omega_t = 1$. Now, by (2.5) we obtain that the polynomial:

$$y = x^2 + \frac{1}{3}\cdot(2+3i) - \frac{2}{3} = x^2 + i$$

is only polynomial solution of the equation (2.6).

Remark. Let $B = B(t)$ be a real polynomial and $c_0, c_1, \ldots, c_{n-1}(c_n = 1)$ be real constants. Let us, similarly as in section 1, denote by $\left[\sqrt[n]{B(t)}\right]$ the real polynomial part of expansion of $\sqrt[n]{B(t)}$ in decreasing powers of t (if n is even, then we suppose that the highest coefficient of the polynomial $B(t)$ is positive). Then, necessary and sufficient conditions for the equation (2.1) to have real polynomial solutions can be obtained as a corollary of theorem **2** an corollary **2**. We considered this case in [4].

We shall prove theorem **2** in next section as a corollary of a more general statement (theorem **3**), which is significant in itself in connection with polnomial solutions of algebraic equations in two variables. We shall prove theorem **1** using theorem **2**.

Proof of theorem **1**. Let $B = B(x)$ be a polynomial, for the degree of which (1.1) holds. Then the condition

$$\begin{cases} \text{there exist complex constants } c_0, c_1, \dots, c_{n-1} \\ \text{and a polynomial } y = y(x) \text{ such that} \\ B(x) = c_0 + c_1 \cdot y + \dots + c_{n-1} \cdot y^{n-1} + y^n, \end{cases} \tag{2.7}$$

is equivalent to the condition:

$$\begin{cases} \text{there exist complex constants } a_0, a_1, \dots, a_{n-2} \\ \text{and a polynomial } v = v(x) \text{ such that} \\ B - a_0 = a_1 \cdot v + a_2 \cdot v^2 + \dots + a_{n-2} \cdot v^{n-2} + v^n. \end{cases} \tag{2.8}$$

The condition (2.8), considering theorem **2**, is equivalent to (for $n \geq 3$) the condition

$$\begin{cases} \text{there exist a number } t \in \{1, 2, \dots, n\} \text{ and a complex constant } c \\ \text{such that the polynommial } Q - c \\ \text{a linear combination of the plynomials } \omega_t S, (\omega_t S)^2, \dots, (\omega_t S)^{n-2} \end{cases}$$

(where S and Q are determined by (1.3)) i.e. to the condition

$$\begin{cases} \text{there exists a complex constant } c \text{ such that the polynomial} \\ Q - c \text{ is a linear combination of the polynomials } S, S^2, \dots, S^{n-2} \end{cases}$$

which, due to linear indepedence of the polynomials $S, S^2, \dots, S^{n-2}$, is equivalent to the condition

$$\begin{cases} \text{there exists a complex comstant } c \text{ such that the polynomials} \\ Q - c, S, S^2, \dots, S^{n-2} \text{are linearly dependent.} \end{cases}$$

This last condition (which is equivalent, to (2.8) and for $n = 2$) is equivalent for instance, to

$$\begin{cases} \text{there exists a complex constant } c, \text{ such that the polynomials} \\ Q - c, S, S^2, \dots, S^{n-2} \text{are linearly dependent on } [0, 1]. \end{cases} \tag{2.9}$$

The polynomials form a unitary space on $[0, 1]$ with the scalar product

$$(A, B) = \int_0^1 A(\tau) \cdot \overline{B(\tau)}\, d\tau$$

from which follows (e.g. [3, pp.207–208]–condition for linear dependence of a finite number of vectors of unitary space) that the condition (2.9) is fulfilled iff there exists a complex constant c such that

$$\begin{vmatrix} \int_0^1 \left(Q(\tau)-c\right)\cdot\left(\overline{Q(\tau)}-\overline{c}\right)\cdot d\tau & \int_0^1 \left(Q(\tau)-c\right)\cdot\overline{S(\tau)}\cdot d\tau & \dots & \int_0^1 \left(Q(\tau)-c\right)\overline{S(\tau)}^{\,n-2}\cdot d\tau \\ \int_0^1 S(\tau)\cdot\left(\overline{Q(\tau)}-\overline{c}\right)\cdot d\tau & \int_0^1 S(\tau)\cdot\overline{S(\tau)}\cdot d\tau & \dots & \int_0^1 S(\tau)\cdot\overline{S(\tau)}^{\,n-2}\cdot d\tau \\ \dots & \dots & \dots & \dots \\ \int_0^1 S^{n-2}(\tau)\cdot\left(\overline{Q(\tau)}-\overline{c}\right)\cdot d\tau & \int_0^1 S^{n-2}(\tau)\cdot\overline{S(\tau)}\cdot d\tau & \dots & \int_0^1 S^{n-2}(\tau)\cdot\overline{S(\tau)}^{\,n-2}\cdot d\tau \end{vmatrix} = 0$$

which, after (1.4), reduces to (1.5). The fact that (2.7) is equivalent to (1.2), completes the proof of the theorem**1**.

3. Let us consider the algebraic equation

$$B_{v_0}(x)\cdot y^{v_0} + B_{v_1}(x)\cdot y^{v_1} + \cdots + B_{v_n}(x)\cdot y^{v_n} = 0$$

$$0 \le v_0 < v_1 < \cdots < v_n, \qquad n \ge 1, \quad v_n \ge 2, \tag{3.1}$$

where $B_{v_k} = B_{v_k}(x)$ $(k = 0, 1, \dots, n)$ is a polynomial of degree b_{v_k}.

Clearly, the degrees of the polynomial solutions of the equation (3.1) can only be the numbers:

$$r = \frac{b_{v_1} - b_{v_j}}{v_j - v_i} \qquad (i < j;\ i, j = 0, 1, \dots, n) \tag{3.2}$$

if they are non negative integers.

For fixed i and j (where $i < j$), for which $b_{v_i} - b_{v_j}$ is a multiple of $q = v_j - v_i$, the polynomials $S = S(x)$ and $Q = Q(x)$ shal be determined as:

$$S = \left[\sqrt[q]{-B_{v_i}/B_{v_j}}\right], \qquad q = v_j - v_i\,, \tag{3.3}$$

$$-B_{v_i} = B_{v_j}\cdot S^q + Q\,. \tag{3.4}$$

These polynomials have been introduced in [5] to describe some polynomial solutions of Riccati–type algebraic differential equations. It can be shown that ([1], [5] or more fully [2. pp. 82-83]):

$$\deg Q < b_{v_j} + (q-1)\deg S\,, \qquad q = v_j - v_i\,; \tag{3.5}$$

and also

$$\left\{\begin{array}{l}\text{the pair } (S, Q)\text{, with } S \text{ taken up to a factor of a } q\text{–th root unity,}\\ \text{is the only pair of polynomials for which (3.4) and (3.5) hold.}\end{array}\right. \tag{3.6}$$

Following the procedure of [5], we shall prove the next theorem.

Theorem **3**. Let $i < j$, $0 \le i, j \le n$ and let $b_{v_i} - b_{v_j}$ be a multiple of $q = v_j - v_i$. If the coefficient conditions hold:

$$b_{v_k} < \frac{(v_j - v_k - 1)b_{v_i} - (v_i - v_k - 1)b_{v_j}}{v_j - v_i} \quad (k=0, 1, \ldots, n;\ k \ne i, j), \tag{3.7}$$

then the equation (3.1) has polynomial solutions of degree (3.2) iff there exists a number $t \in \{1, 2, \ldots, q\}$ such that

$$Q \cdot (\omega_t \cdot S)^{v_i} = \sum_{\substack{k=0 \\ k \ne i,\, j}}^{n} B_{v_k} \cdot (\omega_t \cdot S)^{v_k}, \tag{3.8}$$

where the polynomials S and Q are determined by (3.3) and (3.4) and $\omega_1, \omega_2, \ldots, \omega_q$ are the q^{th} roots of 1; if (3.8) holds for same $t = t_0 (1 \le t_0 \le q)$, then the polynomial $y = \omega_{t_0} \cdot S$ is a solution of (3.1) and the equation (4.1) cannot have polynomial solutions of degree (3.2), other than the function $y = \omega_t \cdot S$, $t = 1, 2, \ldots, q$.

Proof. Assuming that the coefficient conditions (3.7) hold, we determine polynomials S and Q by (3.3) and (3.4). Because of (3.4), we can write the equation (3.1) as:

$$B_{v_j} \cdot (y^{v_j} - S^{v_j - v_i} \cdot y^{v_i}) = Q \cdot y^{v_i} = - \sum_{\substack{k=0 \\ k \ne i,\, j}}^{n} B_{v_k} \cdot y^{v_k}. \tag{3.9}$$

Let the polynomial $y = y(x)$ of degree r, specified by (3.2), be a solution of the equation (3.1). Using (3.5) and (3.7) for $k \ne i, j$, we obtain easily

$$\deg\,(B_{v_k} \cdot y^{v_k}),\ \deg\,(Q \cdot y^{v_i}) < (v_j - 1) \cdot r + b_{v_j}$$

after which, considering that (3.9) is assumed to be an identity assuming further that $y = a_r \cdot x^r + a_{r-1} \cdot x^{r-1} + \cdots + a_0$, $S = s_r \cdot x^r + s_{r-1} \cdot x^{r-1} + \cdots + s_0$, and equating the coefficients of terms with degree $v_j \cdot r,\ v_j \cdot r - 1, \ldots, v_j \cdot r - r$ of y^{v_j} and $S^{v_j - v_i} \cdot y^{v_i}$, we obtain by elementary calculation that $y = \omega_t \cdot S$ for some $t \in \{1, 2, \ldots q\}$. Hence, substituting into (3.9), the condition (3.8) follows directly. Conversaly, if the condition (3.8) holds for same $t \in \{1, 2, \ldots, q\}$ then, because of the fact that, by (3.4),

$$B_{v_j} \cdot (\omega_t \cdot S)^{v_j} + B_{v_i} \cdot (\omega_t \cdot S)^{v_i} = -Q \cdot (\omega_t \cdot S)^{v_i} ,$$

it immediately follows that $y = \omega_t \cdot S$ is a solution of (3.1). Taking (3.6) into consideration completes the proof.

Proof of theorem **2**. Let us apply theorem **3** to the equation (2.2), taking $v_i = v_0 = 0$, $v_j = v_{n-1} = n$ (here $v_k = k$, $k = 1, 2, \dots, n-2$). The coefficient conditions (3.7) are trivially fulfilled in this case. Clearly, the equation (2.2) can have polynomial solutions of degree $m = \deg B/n$ only. In this case the polynomials S and Q are determined by (1.3) ($B_{v_i} = B_0 = -B$, $B_{v_j} = B_n = 1$, $v_j - v_i = n$) because of which (3.8) reduces to (2.3).

References

[1] Bhargava, M., Kaufman, H.: *Collect. Math.*, v. **17**, p. 135–143, 1965

[2] Горбузов, В., Н., Самодуров, А., А.: *Уравнения Риккати и Абеля* Гродно, пп. 101, 1986

[3] Дочев, К., Димитров, Д.: *Линейна алгебра*, София 1977

[4] Лазов, Р., П.: Зб. тр. Ел. фак., Скопје, Вол. **14**, т.8, с. 33–40, 1991

[5] Орещенко, Л., Г., *Диффер. Уравн.*, т.10, N 2, ц.253–257, 1974

[6] Rainville, E., D., Amer. Math. Monthly, v. **43**, p. 473–476, 1936

КРИТЕРИУМ ЗА ДЕКОМПОЗИЦИЈА НА ПОЛИНОМИ

Петар П. Лазов

Р е з и м е

Нека $B = B(x)$ е комплексен полином за чиј степен важи:

$$st\, B(x) = m \cdot n, \qquad m \geq 1, \quad n \geq 2, \quad m, n \in N.$$

Во работата е добиен критериум (теорема **1**) за декомпозиција на полиномот $B(x)$ т.е. за важење на следниот резултат

$$\begin{cases} \text{постојат комплексни полиноми } y = y(x) \text{ со } \mathrm{st}y\,(x) = m \text{ и} \\ u = u(x) \text{ со } \mathrm{st}u\,(x) = n \text{ такви што } B(x) = u\big(y(x)\big). \end{cases}$$

Критериумот е ефективно проверлив. Помошниот резултат што притоа се користи (теорема **2**, односно последица **2**), од своја страна, комплетно го решава проблемот за полномошните решенија на алгебарската равенка

$$B(x) = c_0 + c_1 \cdot y + \cdots + c_{n-1} \cdot y^{n-1} + c_n \cdot y^n\,,$$

вклучувајќи и алгоритми за нивно наоѓање. Теоремата **2** е специјален случај на многу поопштото тврдење (теорема **3**) за алгебарските равенки од две променливи, докажано во полседната точка од работата.

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