EQUIVALENCE OF THE INVERSE SYSTEM APPROACH AND THE INTRINSIC APPROACH TO PROPER SHAPE

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Abstract. There are several equivalent approaches to proper shape theory. Only the original approach of Ball and Sher, defined in [4], induces a category which is a subcategory of the category obtained by other approaches. In [1] it is proved that the proper shape category of Ball and Sher, defined in [4] is a subcategory of the category of proper shape obtained by proper ANR expansions. The objects of the category are locally compact, separable and metrizable spaces embedded in Hilbert cube without a point.

In [2] is given an intrinic definition of proper shape by use of proper proximate nets, i.e. the morphisms are homotopy classes of proper proximative nets.
In the paper it is proved that the proper shape category obtained by the intrinsic approach from [2] is isomorphic to the category obtained by ANR expansions.

0. Proper $\mathcal{V}$-continuous functions

All spaces are locally compact, separable, metrizable. A function is a map which must not be continuous.

Definition: A function $f : X \to Y$ is proper if: for any compact subset D of Y there exists a compact C in X such that $f(X \setminus C) \subseteq Y \setminus D$.

Proposition 1. If $f : X \to Y$ is a proper function and $P \subseteq X$ is closed then $f|_P : P \to Y$ is proper.

Proof. Let $D \subseteq Y$ be compacr. Since f is proper, there exists a compact set $C \subseteq X$ such that $f(X \setminus C) \subseteq Y \setminus D$. The set $C \cap P$ is compact, since it is closed in C.

From, $f|_P (P \setminus (C \cap P)) = f|_P (P \setminus C) \subseteq f(X \setminus C) \subseteq Y \setminus D$, it follows that $f|_P$.

Proposition 2. Let $Y' \subseteq Y$ be closed. Then the inclusion $i : Y' \to Y$ is proper.

Proof. By Proposition 1, the function i is proper as a restriction of 1_Y.

Proposition 3. Let $f : X \to Y$ and $g : Y \to Z$ be proper. Then $gf : X \to Z$ is proper.

Proof. Let $D \subseteq Z$ be compact. There exist compact sets $C \subseteq Y$ and $A \subseteq X$ such rhat

$$g\left(Y \setminus C\right) \subseteq Z \setminus D \qquad (1)$$

$$f\left(X \setminus A\right) \subseteq Y \setminus C \,. \qquad (2)$$

Then $x \in X \setminus A \overset{(1)}{\Rightarrow} f(x) \in Y \setminus C \overset{(2)}{\Rightarrow} g\left(f(x)\right) \in Z \setminus D$.

It follows $(gf)\left(X \setminus A\right) \subseteq Z \setminus C$ and gf is proper.

Proposition 4. Let $f : X \to Y$ be a proper function and $1Q \subseteq Y$ be a set such that $f(X) \subseteq Q$. Let $f' : X \to Q$ be defined by $f'(x) = f(x)$, for every $x \in X$. Then f' is a proper function.

Proof. Let $D \subseteq Q$, D compact. Then. There exists a compact C in X such that $f\left(X \setminus C\right) \subseteq Y \setminus D$. Since $f(X) \subseteq Q$ it follows that $f'\left(X \setminus C\right) \subseteq Q \setminus D$. So, f' is a proper function.

Definition. Two ***continuous proper maps*** $f, g : X \to Y$ are ***properly homotopic*** ($f \simeq_p g$), if there exists a continuous proper map $H : X \times I \to Y$ such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

The class of f we denote by $[f]_p$, and we denote by $[X,Y]_p$ the set of all classas of continuous proper maps from X to Y.

For two families of subsets of X, $\mathcal{A}$ and $\mathcal{B}$, the notations $\mathcal{A} \prec \mathcal{B}$ means that $\mathcal{A}$ is finer than $\mathcal{B}$. By $\operatorname{cov}(X)$ we denote the set of all star-finite coverings of X consisting of open sets with compact closure (= relatively compact sets). The set $\operatorname{cov}(X)$ is ordered i.e. $\mathcal{U} \le \mathcal{V}$ iff $\mathcal{V} \prec \mathcal{U}$.

If $\mathcal{U} \in \operatorname{cov}(X)$, $A \subseteq X$, then $\operatorname{st}(A,\mathcal{U}) = \cup\left\{U \in \mathcal{U} \middle| A \cap U \ne \varnothing\right\}$ and $\operatorname{st}\mathcal{U} = \left\{\operatorname{st}(U,\mathcal{U}) \middle| U \in \mathcal{U}\right\}$.

Definition. Let $\mathcal{V} \in \operatorname{cov}(Y)$. The functions $f, g : X \to Y$ are $\mathcal{V}$-close ($f \overset{\mathcal{V}}{=} g$) if fo any $x \in X$ there exists $V \in \mathcal{V}$ such that $f(x), g(x) \in \mathcal{V}$ (i.e. if $\left\{\{f(x), g(x)\} \middle| x \in X\right\} \prec \mathcal{V}$).

Proposition 5. If for some $\mathcal{V} \in \operatorname{cov}(Y)$, the function $f : X \to Y$ is $\mathcal{V}$- near to a proper function $g : X \to Y$, then $f : X \to Y$ is proper.

Proof. [2] Lemma 3.1.

Let $\mathcal{V} \in \operatorname{cov}(Y)$.

Definition. The function $f : X \to Y$ is $\mathcal{V}$***-continuous***, if any $x \in X$ has a neighbourhood U such that $f(U)$ is a subset of some member of $\mathcal{V}$. With different words, there exists $\mathcal{U} \in \operatorname{cov}(X)$ such that $f(\mathcal{U}) \prec \mathcal{V}$, i.e. $\mathcal{U} \prec f^{-1}(\mathcal{V})$.

Clearly, f is continuous iff f is $\mathcal{V}$- continuous, for every $\mathcal{V} \in \operatorname{cov}(Y)$.

Definition. Let $f, g : X \to Y$ be proper $\mathcal{V}$- continuous functions. Functions f, g are ***properly*** $\mathcal{V}$ ***-continuous homotopic*** **(** $f \overset{\mathcal{V}}{\sim}_p g$ **)**, if there exists a $\operatorname{st}\mathcal{V}$-continuous function $H : X \times I \to Y$, such that $H_0 = f$, $H_1 = g$ and $H|_W$ is $\mathcal{V}$-continuous for some neighbourhood W of $X \times \{0,1\}$ in $X \times I$ (i.e. for any $x \in X$ there exist a neighbourhood U and $\varepsilon > 0$, and $W_1, W_2 \in \mathcal{V}$ such that
$H(U \times [0,\varepsilon)) \subseteq W_1, H(U \times (1-\varepsilon, 1]) \subseteq W_2$).

We say that the function H is a ***proper*** $\mathcal{V}$ - ***continuous homotopy*** from f to g.

Proposition 6. The relation $\overset{\mathcal{V}}{\sim}_p$ is an equivalence relation.

Example. Let $Y = [0,1]$, $\mathcal{V} = \left\{\left[0, \frac{2}{3}\right), \left(\frac{1}{3}, 1\right]\right\} \in \operatorname{cov}(Y)$, $F, G : X \times I \to Y$ be defined by $F(x,t) = \begin{cases} 0, t < 1 \\ \frac{1}{2}, t = 1 \end{cases}$ and $G(x,t) = \begin{cases} \frac{1}{2}, t = 0 \\ 1, t > 0 \end{cases}$. The function $H : X \times I \to Y$ is defined by $H(x,t) = \begin{cases} F(x, 2t), t \le \frac{1}{2} \\ G(x, 2t-1), t \ge \frac{1}{2} \end{cases}$. Then H is not $\mathcal{V}$- continuous, while H is $\operatorname{st}\mathcal{V}$-continuous.

Lemma 0.1. For two $\mathcal{V}$- continuous functions $f, g : X \to Y$ we have: $f \overset{\mathcal{V}}{=} g \Rightarrow f \overset{\mathcal{V}}{\sim}_p g$.

Proof. [2] Lemma 3.2.

Lemma 0.2. Let $\mathcal{V} \in \operatorname{cov}(Y)$ and $\mathcal{W} \in \operatorname{cov}(Z)$, and $h : Y \to Z$ be a function such that $h(\mathcal{V}) \prec \mathcal{W}$. Then , from $f \overset{\mathcal{V}}{\sim}_p g$ it follows $hf \overset{\mathcal{W}}{\sim}_p hg$, for any two $\mathcal{V}$-continuous functions $f, g : X \to Y$.

Proof. [2] Lemma 3.3.

Lemma 0.3. Let $\mathcal{V} \in \operatorname{cov}(Y)$ and $f, g : X \to Y$ be two $\mathcal{V}$-continuous functions. If $f \overset{\mathcal{V}}{\sim}_p g$ then there exists $\mathcal{U} \in \operatorname{cov}(X)$ such that $f(\mathcal{U}), g(\mathcal{U}) \prec \mathcal{V}$ and $fh \overset{\mathcal{V}}{\sim}_p gh$, for any proper $\mathcal{U}$- continuous functions $h : Z \to X$, where Z is an arbitrary space.

Proof. [2] Lemma 3.4.

1. Proper proximative nets

Definition. Proper proximative net (PPN) $(f_\lambda) : X \to Y$ is a net of proper functions $f_\lambda : X \to Y$ indexed by directed set $\Lambda = (\Lambda, \leq)$, such that for any $\mathcal{V} \in \operatorname{cov}(Y)$ there exists $\lambda_0 \in \Lambda$ such that $f_{\lambda_0} \overset{\mathcal{V}}{\sim}_p f_\lambda$, for all $\lambda \geq \lambda_0$, and f_λ is $\mathcal{V}$-continuous for all $\lambda \geq \lambda_0$.

Definition. Two proper proximative nets (f_λ) and (g_δ) (индексирана со подреденото множество Δ) are **properly homotopic,** if for any $\mathcal{V} \in \operatorname{cov}(Y)$ there exist $\lambda_0 \in \Lambda, \delta_0 \in \Delta$ such that $f_\lambda \overset{\mathcal{V}}{\sim}_p g_\delta$, for all $\lambda \geq \lambda_0, \delta \geq \delta_0$. We denote this relation by $(f_\lambda) \simeq_p (g_\delta)$, and this is an equivalence relation..

The composition of $(f_\lambda) = \{f_\lambda | \lambda \in \Lambda\} : X \to Y$ and $(g_\delta) = \{g_\delta | \delta \in \Delta\} : Y \to Z$ we define by $(g_\delta f_\lambda) = \{g_\delta f_\lambda | (\lambda, \delta) \in \Lambda \times \Delta\}$.That the composition is PPN follows from lemma 0.2 and lemm 0.3.

The class of proper homotopy of (f_λ) we denote by $[(f_\lambda)]_p$.

The set of all classes of proper proximative nets from X to Y we denote by $[X, Y]_{ppn}$.

In this way we obtained a category P_p of locally compact, separable metric spaces and proper homotopy classes of proper proximative nets ([2], pp. 307-308).

2. The category Sh^p

Definition. Proper ANR expansion of a locally compact meric space X consists of an inverse system $\underline{X} = \left(X_a, [f_{aa'}]_p, A\right)$ of locally compact ANR's and of a morphism $\underline{f} : X \to \underline{X}$ in pro-PH, i.e. $\underline{f} = \left([f_a]_p, a \in A\right)$, such that

1) If P is a locally compact ANR and $h : X \to P$ is a proper continuous function, then there exists $a \in A$ and a proper continuous function $h_a : X_a \to P$, such that $h_a f_a \simeq_p h$;

2) Let P be a locally compact ANR. let $a \in A$ and let $h_a, h_a' : X_a \to P$ be proper continuous functions such that $h_a f_a \simeq_p h_a' f_a$. Then, there exists $a' \geq a$ such that $h_a f_{aa'} \simeq_p h_a' f_{aa'}$.

Theorem 1. Let $\underline{p} : X \to \underline{X}$, $\underline{X} = \left(X_a, [p_{aa'}]_p, A\right)$ be a proper ANR expansion of X and let $\underline{Y} = \left(Y_b, [q_{bb'}]_p, B\right)$ be an inverse system of locally compact ANR's. If $\underline{g} : X \to \underline{Y}$ is a morphism in pro-PH, then there exists an unique morphism $\underline{F} : \underline{X} \to \underline{Y}$ in pro-PH, such that $\underline{F}\underline{p} = \underline{g}$.

Theorem 2. Let $\underline{p} : X \to \underline{X}$ and $\underline{p}' : X \to \underline{X}'$ be proper ANR expansions. Then there exists an unique morphism $\underline{i} : \underline{X} \to \underline{X}'$ in pro-PH, such that $\underline{i}\,\underline{p} = \underline{p}'$.

The proofs of theorems 1 and 2 are given in [1].

Let X and Y be closed subsets of K. Then there exist ANR expansions of X and Y in K consisting of closed neighbourhood of X and Y (Theorem 2 of [1]). Let $\underline{i} : X \to \underline{X}$, $\underline{X} = \left(X_a, [i_{aa'}]_p, A\right)$ and $\underline{i}' : X \to \underline{X}'$, $\underline{X}' = \left(X_{a_1}, [i_{a_1 a_1'}]_p, A_1\right)$ be two proper ANR expansions of X in K, and let $\underline{i_1} : Y \to \underline{Y}$, $\underline{Y} = \left(Y_b, [i_{bb'}]_p, B\right)$ and $\underline{i_1}' : Y \to \underline{Y}'$, $\underline{Y}' = \left(Y_{b_1}, [i_{b_1 b_1'}]_p, B_1\right)$ be two ANR expansions of Y in K. By theorem 2 there exists an unique isomorphism $\underline{j} : \underline{X} \to \underline{X}'$ such that $\underline{j}\,\underline{i} = \underline{i}'$ and there exists an unique isomorphism $\underline{j}' : \underline{Y} \to \underline{Y}'$ such that $\underline{j}'\underline{i_1} = \underline{i_1}'$.

Let $\underline{F} : \underline{X} \to \underline{Y}$ and $\underline{F}' : \underline{X}' \to \underline{Y}'$ be morphisms in pro-PH. We define a relation „~“ by

$$\left(\underline{i},\underline{i}_1,\underline{F}\right) \sim \left(\underline{i}',\underline{i}_1',\underline{F}'\right) \text{ iff } \underline{j}'\underline{F} = \underline{F}'\underline{j}.$$

Morphisms of proper shape (morphisms of the category Sh^p) are the equivalence classes of this relation, i.e. $\overline{F} = \left[\left(\underline{i},\underline{i}_1,\underline{F}\right)\right]$.

The composition of morphisms $\overline{F} : X \to Y$, $\overline{F} = \left[\left(\underline{i},\underline{i}_1,\underline{F}\right)\right]$ and $\overline{G} : Y \to Z$, $\overline{G} = \left[\left(\underline{i}_1,\underline{i}_2,\underline{G}\right)\right]$ is defined as a class of $\left(i,i_2,GF\right)$.

3. The relation of categories

Lemma 4. Let Y be an ANR for metric spaces. For any $\mathcal{V} \in \operatorname{cov}(Y)$ there exists $\mathcal{V}' \in \operatorname{cov}(Y)$ such that for any metric space X and every function $f \in C_{\mathcal{V}'}(X,Y)$, there exists a continuous function $g : X \to Y$ such that f and g are $\mathcal{V}$ - near.

Proof. It follows from [3] Lemma 1.8. and the fact that in metric spaces any open covering is normal.

Since f is $\mathcal{V}'$-continuous, from the proof of lemma 4, it follows that f is $\mathcal{V}$ - continuous. Then, lemma 0.1 implies that $f \overset{\mathcal{V}}{\sim}_p g$. It follows that: any $\mathcal{V}$ - continuous function is $\mathcal{V}$ -properly homotopic with a continuous function.

Lemma 5. Let Y be an ANR. There exists $\mathcal{V}_Y \in \operatorname{cov} Y$ such that $f \overset{\mathcal{V}_Y}{\sim}_p g$ implies $f \simeq_p g$, for any two continuous functions $f,g : X \to Y$.

Proof. [2] Lemma 3.9.

The following properties can easily be verified:

- The set $\left\{\mathcal{V} \middle| \mathcal{V} \in \operatorname{cov} Y, \mathcal{V} \geq \mathcal{V}_Y\right\}$ is cofinal in $\operatorname{cov} Y$.
- Let $\left(f_\lambda, \lambda \in \mathrm{M}\right)$ be a PPN, and M be cofinal in Λ. Then there exists a unique PPN defined on Λ (up to isomorphism of PPN), which restricted to M is isomorphic to $\left(f_\lambda, \lambda \in \mathrm{M}\right)$ and is isomorphic to $\left(f_\lambda, \lambda \in \mathrm{M}\right)$.

By K we denote Hilbert cube without a point.

Let Y be embedded in K. We define the set $\operatorname{cov}_Y K$ of coverings of Y by

$$\operatorname{cov}_Y K = \{ \mathcal{V} \mid \ \forall V \in \mathcal{V},\ V \cap Y \neq \varnothing \}$$

For any $\mathcal{V} \in \operatorname{cov}_Y K$, we define a closed neighbourhood of Y by

$$Y_{\mathcal{V}} = \overline{\bigcup_{V \in \mathcal{V}} V}$$

The set of closed neighbourhoods of Y, $\{Y_{\mathcal{V}} | \mathcal{V} \in \operatorname{cov}_Y K\}$ is cofinal in the set of all neghbourhoods of Y, and such that $\mathcal{V} \prec \mathcal{V}'$ implies $Y_{\mathcal{V}} \subseteq Y_{\mathcal{V}'}$.

From lemma 5, there exists a covering $\mathcal{V}_K \in \operatorname{cov} K$, such that $f \overset{\mathcal{V}_K}{\sim}_p g$ implies $f \simeq_p g$, for any tho continuous functions $f, g : X \to K$.

Let $\operatorname{cov}^0{}_Y K = \{\mathcal{V} | \mathcal{V} \in \operatorname{cov}_Y K, \mathcal{V} \geq \mathcal{V}_K\}$.

Now, let $(f_\lambda, \lambda \in \Lambda)$ be PPN from X to Y, and Λ cofinite in $\operatorname{cov}^0{}_Y K$. To $(f_\lambda, \lambda \in \Lambda)$, we associate the triple $(\underline{i}, \underline{j}, \underline{F})$ in the following way:

Let $\underline{i} : X \to \underline{X}, \underline{i} = ([i_a]_p | a \in A), \underline{X} = (X_a, [i_{aa'}]_p, A)$ and $\underline{j} : Y \to \underline{Y}, \underline{j} = ([j_b]_p | b \in B), \underline{Y} = (Y_b, [j_{bb'}]_p, B)$ be proper ANR expansions of X and Y, respectively, and B be cofinal in $\operatorname{cov}^0{}_Y K$.

Let $b \in B$. For any f_λ there exists a continuous function $g_\lambda : X \to Y_b$, such that $f_\lambda \overset{\lambda}{\sim}_p g_\lambda$, in Y_b. Moreover, $(f_\lambda) \sim_p (g_\lambda)$, as PPN's from X to K.

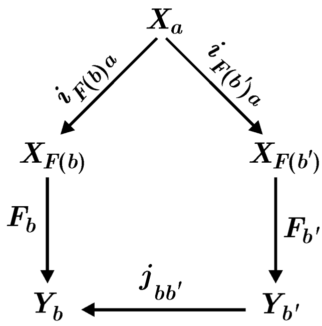
The continuous function g_λ we extend to an open neighbourhood U of X. There exists a closed neighbourhood P of X such that $P \subseteq U$ and $g_{\lambda'}|_P \simeq_p g_\lambda|_P$ in Y_b, for all $\lambda' \geq \lambda$, and there exists $X_a \subseteq P$ such that $g_{\lambda'}|_{X_a} \simeq_p g_\lambda|_{X_a}$ in Y_b, for all $\lambda' \geq \lambda$ (it follows from lemma 1 and lemma 2 of [1]).

We put $F(b) = a$ and $F_b = g_\lambda|_{X_{F(b)}} : X_{F(b)} \to Y_b$.

<u>We will show that</u> $([F_b]_p, F)$ <u>is a morphism in inv-PH</u>**.** Let $b \leq b'$, i.e. $Y_{b'} \subseteq Y_b$. Then $F_b = g_\lambda|_{X_a} : X_{F(b)} \to Y_b$ and $F_{b'} = g_{\lambda'}|_{X_{a'}} : X_{F(b')} \to Y_{b'}$.

There exists $a \in A$ such that $a \geq F(b), F(b')$. Since X_a is closed subset of $X_{F(b)}$, we have $g_\lambda|_{X_a} \simeq_p g_{\lambda_1}|_{X_a}$ in Y_b, for all $\lambda_1 \geq \lambda$. Since X_a is closed subset of $X_{F(b')}$, we have $g_{\lambda'}|_{X_a} \simeq_p g_{\lambda_2}|_{X_a}$ in $Y_{b'}$, for all $\lambda_2 \geq \lambda'$.

There exists $\lambda'' \geq \lambda, \lambda'$. Then $g_\lambda|_{X_a} \simeq_p g_{\lambda''}|_{X_a}$ in Y_b and $g_{\lambda'}|_{X_a} \simeq_p g_{\lambda''}|_{X_a}$ in $Y_{b'}$. It follows $j_{bb'} g_{\lambda'}|_{X_a} \simeq_p g_\lambda|_{X_a}$ in Y_λ, i.e. $j_{bb'} F_{b'} i_{F(b')a} \simeq_p F_b i_{F(b)a}$, so the following diagram commutes up to proper homotopy



It is proved that $\left([F_b]_p, F\right)$ is a morphism in inv-PH. Let $\underline{F}$ be a morphism in pro-PH, defined by (the class of equivalence of) $\left([F_b]_p, F\right)$. To $\left(f_\lambda, \lambda \in \Lambda\right)$ we associate the triple $\left(\underline{i}, \underline{j}, \underline{F}\right)$.

<u>We have to show that the definition doesn't depend on choice of the represent of a class of proper homotopy of PPN.</u> Let $\left(f_\lambda, \lambda \in \Lambda\right)$ and $\left(f'_\delta, \delta \in \Delta\right)$ be two PPN being properly homotopic. Then Λ and Δ are cofinal in $\operatorname{cov}^0{}_Y K$. Let $\underline{i}' : X \to \underline{X}', \underline{i}' = \left\{[i_c]_p \middle| c \in C\right\}, i_c : X \to X_c, \underline{X}' = \left(X_c, [i_{cc'}], C\right)$ and $\underline{j}' : Y \to \underline{Y}', \underline{j}' = \left\{[j_d]_p \middle| d \in D\right\}, j_d : Y \to Y_d, \underline{Y}' = \left(Y_d, [j_{dd'}], D\right)$ be another proper ANR expansions of X and Y respectively. Then D is cofinal in $\operatorname{cov}^0{}_Y K$. From theorem 2, there exists a unique isomorphism in pro-PH, $\underline{k} : \underline{X} \to \underline{X}'$, such that $\underline{k}\,\underline{i} = \underline{i}'$. It is defined in the following way: for $c \in C$ there exists a $k(c) \in A$ such that $X_{k(c)} \subseteq X_c$. Then $\underline{k} : \underline{X} \to \underline{X}'$ is defined as the class of the morphism in inv-PH, $\left([i_{ck(c)}]_p, k\right)$. In a similar way is defined $\underline{l} : \underline{Y} \to \underline{Y}'$ with the property $\underline{l}\,\underline{j} = \underline{j}'$.

Now, let $d \in D$. Since the nets are properly homotopic, it follows that their corresponding continuous functions are homotopic i.e., there exist $\lambda_1 \in \Lambda, \delta_1 \in \Delta$ such that for all $\lambda \geq \lambda_1, \delta \geq \delta_1$,

$$g_\lambda\,|_X \simeq_p g'_\delta\,|_X \text{ in } Y_{l(d)}. \qquad (1)$$

Let $\underline{F} : \underline{X} \to \underline{Y}$ be morphism in pro-PH associated with $\left(f_\lambda, \lambda \in \Lambda\right)$, and $\underline{F}' : \underline{X}' \to \underline{Y}'$ be morphism in pro-PH associated with $\left(f'_\delta, \delta \in \Delta\right)$.

From the definition of $\underline{F}'$ it follows that for $d \in D$ и $F'(d)$ there exists $\delta_0 \in \Delta$ such that $g'_\delta |_{X_{F'(d)}} \simeq_p g'_{\delta_0} |_{X_{F'(d)}}$ in Y_d, for all $\delta \geq \delta_0$ ($\delta_0 \in \Delta$ can be chosen to satisfy $\delta_0 \geq \delta_1$).

From the definition of $\underline{F}$, for $l(d) \in B$, there exist $F(l(d))$ and $\lambda_0 \in \Lambda$ such that $g_\lambda |_{X_{F(l(d))}} \simeq_p g_{\lambda_0} |_{X_{F(l(d))}}$ in $Y_{l(d)}$, for all $\lambda \geq \lambda_0$ ($\lambda_0 \in \Lambda$ can be chosen to satisfy $\lambda_0 \geq \lambda_1$).

It follows that $g'_\delta |_X \simeq_p g'_{\delta_0} |_X$ in Y_d, for all $\delta \geq \delta_0$ and $g_\lambda |_X \simeq_p g_{\lambda_0} |_X$ in $Y_{l(d)}$, for all $\lambda \geq \lambda_0$.

From (1) it follows that $g'_{\delta_0} |_X \simeq_p g'_{\lambda_0} |_X$ in $Y_{l(d)} \subseteq Y_d$, i.e. $F'_d |_X \simeq_p j_{dl(d)} F_{l(d)} |_X$ in Y_d.

We obtained that $F'_d i_{F'(d)k(F'(d))} i_{k(F'(d))} \simeq_p j_{dl(d)} F_{l(d)} i_{F(l(d))}$, i.e. $\underline{F}' \underline{k}\, \underline{i} = \underline{l} \underline{F} \underline{i}$, and it follows $\underline{F}' \underline{k} = \underline{l} \underline{F}$.

We proved that $(\underline{i}, \underline{j}, \underline{F}) \sim (\underline{i}', \underline{j}', \underline{F}')$, i.e. the definition doesn't depend on choice of the represent of a class of proper homotopy of PPN.

<u>We will prove that in this way we obtain a functor from the category</u> P_p <u>to the category</u> Sh^p. Let X, Y, Z be closed subsets of K and let

$$\underline{i} : X \to \underline{X}, \underline{i} = \left\{ \left[i_a \right]_p \middle| a \in A \right\}, \underline{X} = \left(X_a, \left[i_{aa'} \right]_p, A \right)$$
$$\underline{j} : Y \to \underline{Y}, \underline{j} = \left\{ \left[j_b \right]_p \middle| b \in B \right\}, \underline{Y} = \left(Y_b, \left[j_{bb'} \right]_p, B \right)$$
$$\underline{m} : Z \to \underline{Z}, \underline{m} = \left\{ \left[m_c \right]_p \middle| c \in C \right\}, \underline{Z} = \left(Z_c, \left[m_{cc'} \right]_p, C \right)$$

be proper ANR expansions of X, Y, Z, respectively.

Let to $(f_\lambda, \lambda \in \Lambda) : X \to Y$ be associated the triple $(\underline{i}, \underline{j}, \underline{F})$ and to $(f'_\delta, \delta \in \Delta) : Y \to Z$ be associated the triple $(\underline{j}, \underline{m}, \underline{G})$. $\underline{F}$ and $\underline{G}$ are morphisms in pro-PH (i.e. the classes of morphisms $\left(\left[F_b \right]_p, F \right)$ and $\left(\left[G_c \right]_p, G \right)$) and their composition is defined as the class of $\left(\left[G_c F_{G(c)} \right]_p, FG \right)$, where $FG : C \to A$ and $G_c F_{G(c)} : X_{FG(c)} \to Z_c$ for $c \in C$.

Let g_λ and g'_δ be continuous functions associated to f_λ and f'_δ. Let $c \in C$. For the closed neighbourhood Z_c of Z, there exists a closed neighbourhood $Y_{G(c)}$ of Y and $\delta_1 \in \Delta$ such that for $\delta \geq \delta_1$ holds $g'_\delta |_{Y_{G(c)}} \simeq_p g'_{\delta_1} |_{Y_{G(c)}}$ in Z_c, i.e. there exists a proper homotopy $\tilde{G} : Y_{G(c)} \times I \to Z_c$ such that $\tilde{G}(y,0) = g'_\delta |_{Y_{G(c)}} (y)$ and $\tilde{G}(y,1) = g'_{\delta_1} |_{Y_{G(c)}} (y)$.

Similarly, for the closed neighbourhood $Y_{G(c)}$ of Y there exists a closed neighbourhood $X_{FG(c)}$ of X and $\lambda_1 \in \Lambda$ such that for all $\lambda \geq \lambda_1$ holds $g_\lambda |_{X_{FG(c)}} \simeq_p g_{\lambda_1} |_{X_{FG(c)}}$ in $Y_{G(c)}$, i.e. there exists a proper homotopy $\tilde{F} : X_{FG(c)} \times I \to Y_{G(c)}$ such that $\tilde{F}(x,0) = g_\lambda |_{X_{FG(c)}} (x)$ and $\tilde{F}(x,1) = g_{\lambda_1} |_{X_{FG(c)}} (x)$.

Then $g'_\delta |_{Y_{G(c)}} \tilde{F} : X_{FG(c)} \times I \to Z_c$ is a proper homotopy connecting $g'_\delta g_\lambda |_{X_{FG(c)}}$ and $g'_\delta g_{\lambda_1} |_{X_{FG(c)}}$, and $\tilde{G}\left(g_{\lambda_1} |_{X_{FG(c)}} (x), t\right)$ is a proper homotopy connecting $g'_\delta g_{\lambda_1} |_{X_{FG(c)}}$ and $g'_{\delta_1} g_{\lambda_1} |_{X_{FG(c)}}$. It follows that $g'_\delta g_{\lambda_1} |_{X_{FG(c)}} \simeq_p g'_{\delta_1} g_{\lambda_1} |_{X_{FG(c)}}$ in Z_c, for $\lambda \geq \lambda_1$ and $\delta \geq \delta_1$.

We proved that foe any $c \in C$ and Z_c, a closed neihbourhood of Z, there exists $FG(c) \in A$, i.e. a closed neihbourhood $X_{FG(c)}$ of X and there exist λ_1, δ_1 such that for $\lambda \geq \lambda_1$ and $\delta \geq \delta_1$ holds $g'_\delta g_\lambda |_{X_{FG(c)}} \simeq_p g'_{\delta_1} g_{\lambda_1} |_{X_{FG(c)}}$ in Z_c.

We put $H = FG$ и $H_c = g'_{\delta_1} g_{\lambda_1} |_{X_{FG(c)}}$. The class of $\left([H_c]_p, H\right)$ is a morphism in pro-PH associated to the composition of $(f_\lambda, \lambda \in \Lambda)$ and $(f'_\delta, \delta \in \Delta)$.

Since $H_c = g'_{\delta_1} g_{\lambda_1} |_{X_{FG(c)}} = g'_{\delta_1} |_{Y_{G(c)}} g_{\lambda_1} |_{Y_{G(c)}} = G_c F_{G(c)}$, we have $\left([H_c]_p, H\right) = \left([G_c]_p, G\right)\left([F_{G(c)}]_p, F\right)$ in inv-PH. It follows $\underline{H} = \underline{G}\underline{F}$ in pro-PH.

By the previous, $\overline{H} : X \to Z$ is a proper shape morphism in Sh^p and $\overline{H} = [(\underline{i}, \underline{m}, \underline{H})] = [(\underline{i}, \underline{m}, \underline{G}\underline{F})]$. From $\overline{G}\overline{F} = [(\underline{i}, \underline{m}, \underline{G}\underline{F})]$, it follows that $\overline{H} = \overline{G}\overline{F}$ in SH^p.

To PPN $\{1_\lambda | \lambda \in \Lambda\}$ is associated the triple $(\underline{i}, \underline{i}, \underline{1}_X)$, where

$\underline{i}: X \to \underline{X}, \underline{i} = \left(\left[i_a\right]_p, a \in A\right), \underline{X} = \left(X_a, \left[i_{aa'}\right]_p, A\right)$, and $\underline{1}_X$ is the class of $\left(\left[1_{X_a}\right]_p, 1_A\right)$.

We proved that we obtained a functor from the category P_p to the category Sh^p.

<u>The functor is injective</u>. Let

$\underline{i}: X \to \underline{X}, \underline{i} = \left(\left[i_a\right]_p, a \in A\right), \underline{X} = \left(X_a, \left[i_{aa'}\right]_p, A\right)$ and

$\underline{i}': X \to \underline{X}', \underline{i}' = \left(\left[i_b\right]_p, b \in B\right), \underline{X}' = \left(X_b, \left[i_{bb'}\right]_p, B\right)$ are two proper ANR expansions of X, and let $\underline{j}: Y \to \underline{Y}, \underline{j} = \left\{\left[j_c\right]_p \middle| c \in C\right\}, \underline{Y} = \left(Y_c, \left[j_{cc'}\right]_p, C\right)$ and

$\underline{j}': Y \to \underline{Y}', \underline{j}' = \left\{\left[j_d\right]_p \middle| d \in D\right\}, \underline{Y}' = \left(Y_d, \left[j_{dd'}\right]_p, D\right)$ be two proper ANR expansions of Y.

Let $\underline{f} = \left\{f_\lambda \middle| \lambda \in \Lambda\right\}$ and $\underline{f}' = \left\{f'_\delta \middle| \delta \in \Delta\right\}$ be two PPNs and let to $\underline{f}$ be asociated $\overline{F} = \left(\underline{i}, \underline{j}, \underline{F}\right)$ and to $\underline{f}'$ be associated $\overline{G} = \left(\underline{i}', \underline{j}', \underline{G}\right)$. Suppose that $\overline{F} = \overline{G}$. Then $\underline{l}\,\underline{F} = \underline{G}\,\underline{k}$, where $\underline{k}: \underline{X} \to \underline{X}'$ and $\underline{l}: \underline{Y} \to \underline{Y}'$ are isomorphisms in pro-PH defined by $\left(\left[i_{bk(b)}\right]_p, k\right)$ and $\left(\left[j_{dl(d)}\right]_p, l\right)$, respectively. Let g_λ and g'_δ be continuous maps associated to f_λ and f'_δ, respectively.

Let $d \in D$ and Y_d be a closed neighbourhood of Y. Then, the map $j_{dl(d)}F_{l(d)}: X_{F(l(d))} \to Y_d$ satisfies

$$j_{dl(d)}F_{l(d)} \simeq_p j_{dl(d)}g_\lambda |_{X_{F(l(d))}} \text{ in } Y_d \text{ for all } \lambda \geq \lambda_0. \qquad (1)$$

For $G_d i_{G(d)k(G(d))}: X_{k(G(d))} \to Y_d$ we have

$$G_d i_{G(d)k(G(d))} \simeq_p g'_\delta i_{G(d)k(G(d))} |_{X_{k(G(d))}} \text{ in } Y_d \text{ for } \delta \geq \delta_0 \qquad (2)$$

From $\underline{l}\,\underline{F} = \underline{G}\,\underline{k}$ it follows that: for $d \in D$ there exists $a \in A$ such that $a \geq F\left(l(d)\right), k\left(G(d)\right)$ and such that

$$j_{d(l(d))}F_{l(d)} |_{X_a} \simeq_p G_d i_{G(d)k(G(d))} | X_a \text{ in } Y_d \text{ for } \lambda \geq \lambda_0. \qquad (3)$$

From (1) and (2) it follows that

$$j_{dl(d)}F_{l(d)} |_{X_a} \simeq_p j_{dl(d)}g_\lambda |_{X_a} \text{ in } Y_d \text{ for all } \lambda \geq \lambda_0. \quad (4)$$

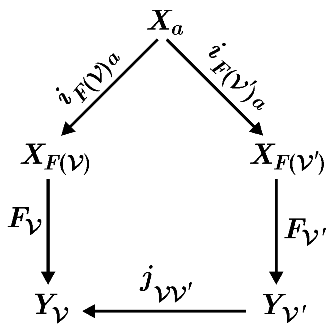
and $G_d i_{G(d)k(G(d))} |_{X_a} \simeq_p g'_\delta i_{G(d)k(G(d))} |_{X_a}$ in Y_d for $\delta \geq \delta_0$ (5)

From (3), (4) and (5) it follows that $j_{d(l(d))}g_\lambda\,|_{X_a} \simeq_p g'_\delta i_{G(d)k(G(d))}\,|_{X_a}$ in Y_d holds, for $\lambda \geq \lambda_0, \delta \geq \delta_0$. Since, homotopy is continuous it follows that $\underline{f} = \left\{ f_\lambda \middle| \lambda \in \Lambda \right\}$ and $\underline{f}' = \left\{ f'_\delta \middle| \delta \in \Delta \right\}$ are properly homotopic, i.e. the functor is an injection.

<u>The functor is surjective</u> . Let $\left(\underline{i}, \underline{j}, \underline{F}\right)$ be a represent of the class $\left[\left(\underline{i}, \underline{j}, \underline{F}\right)\right]$, where $\underline{i} : X \to \underline{X}, \underline{i} = \left(\left[i_a\right]_p, a \in A\right), \underline{X} = \left(X_a, \left[i_{aa'}\right]_p, A\right)$ and

$\underline{j} : Y \to \underline{Y}$, $\underline{j} = \left\{ [j_{\mathcal{V}}]_p \mid V \in \mathcal{V} \right\}$, $\underline{Y} = (Y_v\ , [j_{\mathcal{V}\mathcal{V}'}]_p\ , \mathcal{V} \in \operatorname{cov}^0_K Y)$.

The morphism $\underline{F} : \underline{X} \to \underline{Y}$ consists of a function $F : \operatorname{cov}^0_K Y \to A$ and for each $\mathcal{V} \in \operatorname{cov}^0_K Y$ of a morphism (i.e. a continuous map) $F_{\mathcal{V}} : X_{F(\mathcal{V})} \to Y_{\mathcal{V}}$ such that for all $\mathcal{V}, \mathcal{V}' \in \operatorname{cov}^0_K Y$, $\mathcal{V} \leq \mathcal{V}'$ there exists $a \in A$, $a \geq F(\mathcal{V}), F(\mathcal{V}')$ such that the following diagram commutes up to proper homotopy



We conclude that $F_{\mathcal{V}}\,|_X \simeq_p F_{\mathcal{V}'}\,|_X$ in $Y_{\mathcal{V}}$ for all $\mathcal{V} \leq \mathcal{V}'$.

For $\mathcal{V} \in \operatorname{cov}^0_K Y$ we define the function $f_{\mathcal{V}} : X \to Y$ in the following way: Let $x \in X$.

1) $f_{\mathcal{V}}(x) = F_{\mathcal{V}}(x)$ if $F_{\mathcal{V}}(x) \in Y$

2) If $F_{\mathcal{V}}(x) \notin Y$, then $F_{\mathcal{V}}(x)$ belongs to some element V_x of the covering $\mathcal{V}$ There exists a $y_x \in Y \cap V_x$. We put $f_{\mathcal{V}}(x) = y_x$,

By the construction of $f_{\mathcal{V}}$ it is clear that $f_{\mathcal{V}}$ and $F_{\mathcal{V}}\,|_X$ are $\mathcal{V}$- near. It follows, they are $\mathcal{V}$-properly homotopic in $Y_{\mathcal{V}}$ ($f_{\mathcal{V}}$ is proper since it is $\mathcal{V}$- near to the proper map $F_{\mathcal{V}}\,|_X$).

We denote $\mathcal{V} \cap Y = \{Y \cap V \mid V \in \mathcal{V}\}$

Let w be an arbitrary covering of Y. There exists $\mathcal{V} \in \operatorname{cov}^0_K Y$ such that $\mathcal{W} \leq \mathcal{V}$.

Then $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p F_{\mathcal{V}}|_X \overset{\mathcal{V}}{\sim}_p F_{\mathcal{V}'}|_X \overset{\mathcal{V}}{\sim}_p f_{\mathcal{V}'}$ in $Y_{\mathcal{V}}$ for all $\mathcal{V}' \geq \mathcal{V}$. Since $\mathcal{V} \cap Y \geq \mathcal{W}$, it follows that $f_{\mathcal{V}}$ and $f_{\mathcal{V}'}$ are $\mathcal{V}$-properly homotopic. We conclude that $\left\{ f_{\mathcal{V}} \middle| \mathcal{V} \in \operatorname{cov}_K^0 Y \right\}$ is PPN from X to Y.

Let $g_{\mathcal{V}}$ be a continuous map associated to $f_{\mathcal{V}}$. Then $f_{\mathcal{V}} \overset{\mathcal{V}}{\sim}_p g_{\mathcal{V}}|_X$ in $Y_{\mathcal{V}}$. Since continuous maps $g_{\mathcal{V}}|_X$ and $F_{\mathcal{V}}|_X$ are $\mathcal{V}$- near, from the definition of $\operatorname{cov}^0{}_Y K$ and. lemma 5, it follows that $g_{\mathcal{V}}|_X \simeq_p F_{\mathcal{V}}|_X$ in $Y_{\mathcal{V}}$. From lemma 2 of [1] it follows that there exists a closed neighbourhood P of X such that $g_{\mathcal{V}}|_P \simeq_p F_{\mathcal{V}}|_P$ in $Y_{\mathcal{V}}$. Ther exists $X_a \subseteq P$, so $g_{\mathcal{V}}|_{X_a} \simeq_p F_{\mathcal{V}}|_{X_a}$ in $Y_{\mathcal{V}}$. We conclude that the image of $\left[\left\{ f_{\mathcal{V}} \middle| \mathcal{V} \in \operatorname{cov}_K^0 Y \right\} \right]$ under the functor defined above is $\left[\left(\underline{i}, \underline{j}, \underline{F} \right) \right]$. We proved that the functor is surjective.

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ЕКВИВАЛЕНЦИЈА НА ДЕФИНИЦИЈАТА СО ИНВЕРЗНИ СИСТЕМИ И ВНАТРЕШНАТА ДЕФИНИЦИЈА ВО ТЕОРИЈАТА НА ПРАВ ОБЛИК

Никита Шекутковки, Ѓорѓи Маркоски

Апстракт. Постојат неколку еквивалентни пристапи во теоријата на прав облик. Само оригиналниот пристап на Ball и Sher дефиниран во [4] индуцира категорија која е поткатегорија од категоријата добиена од другите пристапи. Во [1] е докажано дека категоријата на прав облик на Ball и Sher дефинирана во [4] е поткатегорија од категоријата на прав облик добиена со прави ANR експанзии. Објекти во оваа категорија се локално компактни, сепарабилни и метризабилни простори вложени во Хилбертовиот куб без една точка.

Во [2] дадена е внатрешната дефиниција на прав облик користејќи прави проксимативни мрежи, т.е. морфизми се класите не хомотопија на прави проксимативни мрежи.

Во овој труд докажано е дека категоријата на прав облик добиена со внатрепната дефиниција од [2] е изоморфна со категоријата добиена со прави ANR експанзии.