

INEQUALITIES FOR HYPO- q -NORMS ON A CARTESIAN PRODUCT OF INNER PRODUCT SPACES

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Abstract. In this paper we introduce the hypo- q -norms on a Cartesian product of inner product spaces. A representation of these norms in terms of inner products, the equivalence with the q -norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al., Grüss type inequalities, Boas-Bellman and Bombieri type inequalities are also given.

1. INTRODUCTION

Let $(E, \|\cdot\|)$ be a normed linear space over the real or complex number field \mathbb{K} . On \mathbb{K}^n endowed with the canonical linear structure we consider a norm $\|\cdot\|_n$ and the unit ball

$$\mathbb{B}(\|\cdot\|_n) := \{\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \mid \|\boldsymbol{\lambda}\|_n \leq 1\}.$$

As an example of such norms we should mention the usual p -norms

$$\|\boldsymbol{\lambda}\|_{n,p} := \begin{cases} \max\{|\lambda_1|, \dots, |\lambda_n|\} & \text{if } p = \infty; \\ (\sum_{k=1}^n |\lambda_k|^p)^{\frac{1}{p}} & \text{if } p \in [1, \infty). \end{cases} \quad (1.1)$$

The *Euclidean norm* is obtained for $p = 2$, i.e.,

$$\|\boldsymbol{\lambda}\|_{n,2} = \left(\sum_{k=1}^n |\lambda_k|^2 \right)^{\frac{1}{2}}.$$

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It is well known that on $E^n := E \times \cdots \times E$ endowed with the canonical linear structure we can define the following p -norms:

$$\|\mathbf{x}\|_{n,p} := \begin{cases} \max \{\|x_1\|, \dots, \|x_n\|\} & \text{if } p = \infty; \\ (\sum_{k=1}^n \|x_k\|^p)^{\frac{1}{p}} & \text{if } p \in [1, \infty); \end{cases} \quad (1.2)$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

Following [9], for a given norm $\|\cdot\|_n$ on \mathbb{K}^n , we define the functional $\|\cdot\|_{h,n} : E^n \rightarrow [0, \infty)$ given by

$$\|\mathbf{x}\|_{h,n} := \sup_{\lambda \in B(\|\cdot\|_n)} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \quad (1.3)$$

where $\mathbf{x} = (x_1, \dots, x_n) \in E^n$.

It is easy to see, by the properties of the norm $\|\cdot\|$, that:

- (i) $\|\mathbf{x}\|_{h,n} \geq 0$ for any $\mathbf{x} \in E^n$;
- (ii) $\|\mathbf{x} + \mathbf{y}\|_{h,n} \leq \|\mathbf{x}\|_{h,n} + \|\mathbf{y}\|_{h,n}$ for any $\mathbf{x}, \mathbf{y} \in E^n$;
- (iii) $\|\alpha \mathbf{x}\|_{h,n} = |\alpha| \|\mathbf{x}\|_{h,n}$ for each $\alpha \in \mathbb{K}$ and $\mathbf{x} \in E^n$;

and therefore $\|\cdot\|_{h,n}$ is a *semi-norm* on E^n . This will be called the *hypo-semi-norm* generated by the norm $\|\cdot\|_n$ on E^n .

We observe that $\|\mathbf{x}\|_{h,n} = 0$ if and only if $\sum_{j=1}^n \lambda_j x_j = 0$ for any $(\lambda_1, \dots, \lambda_n) \in B(\|\cdot\|_n)$. If there exists $\lambda_1^0, \dots, \lambda_n^0 \neq 0$ such that $(\lambda_1^0, 0, \dots, 0)$, $(0, \lambda_2^0, \dots, 0)$, \dots , $(0, 0, \dots, \lambda_n^0) \in B(\|\cdot\|_n)$ then the semi-norm generated by $\|\cdot\|_n$ is a *norm* on E^n .

If by $\mathbb{B}_{n,p}$ with $p \in [1, \infty]$ we denote the balls generated by the p -norms $\|\cdot\|_{n,p}$ on \mathbb{K}^n , then we can obtain the following *hypo- q -norms* on E^n :

$$\|\mathbf{x}\|_{h,n,q} := \sup_{\lambda \in \mathbb{B}_{n,p}} \left\| \sum_{j=1}^n \lambda_j x_j \right\|, \quad (1.4)$$

with $q > 1$ and $\frac{1}{q} + \frac{1}{p} = 1$ if $p > 1$, $q = 1$ if $p = \infty$ and $q = \infty$ if $p = 1$.

For $p = 2$, we have the Euclidean ball in \mathbb{K}^n , which we denote by \mathbb{B}_n , $\mathbb{B}_n = \left\{ \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathbb{K}^n \mid \sum_{i=1}^n |\lambda_i|^2 \leq 1 \right\}$ that generates the *hypo-Euclidean norm* on E^n , i.e.,

$$\|\mathbf{x}\|_{h,e} := \sup_{\lambda \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \quad (1.5)$$

Moreover, if $E = H$, H is a inner product space over \mathbb{K} , then the *hypo-Euclidean norm* on H^n will be denoted simply by

$$\|\mathbf{x}\|_e := \sup_{\lambda_j \in \mathbb{B}_n} \left\| \sum_{j=1}^n \lambda_j x_j \right\|. \quad (1.6)$$

Let $(H; \langle \cdot, \cdot \rangle)$ be a Hilbert space over \mathbb{K} and $n \in \mathbb{N}$, $n \geq 1$. In the Cartesian product $H^n := H \times \cdots \times H$, for the n -tuples of vectors $\mathbf{x} = (x_1, \dots, x_n)$, $\mathbf{y} = (y_1, \dots, y_n) \in H^n$, we can define the inner product $\langle \cdot, \cdot \rangle$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle := \sum_{j=1}^n \langle x_j, y_j \rangle, \quad \mathbf{x}, \mathbf{y} \in H^n, \quad (1.7)$$

which generates the Euclidean norm $\|\cdot\|_2$ on H^n , i.e.,

$$\|\mathbf{x}\|_2 := \left(\sum_{j=1}^n \|x_j\|^2 \right)^{\frac{1}{2}}, \quad \mathbf{x} \in H^n. \quad (1.8)$$

The following result established in [9] connects the usual Euclidean norm $\|\cdot\|_2$ with the hypo-Euclidean norm $\|\cdot\|_e$.

Theorem 1 (Dragomir, 2007, [9]). *For any $\mathbf{x} \in H^n$ we have the inequalities*

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_e \leq \|\mathbf{x}\|_2, \quad (1.9)$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_e$ are equivalent norms on H^n .

The following representation result for the hypo-Euclidean norm plays a key role in obtaining various bounds for this norm:

Theorem 2 (Dragomir, 2007, [9]). *For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$, we have*

$$\|\mathbf{x}\|_e = \sup_{\|y\| \leq 1} \left(\sum_{j=1}^n |\langle x_j, y \rangle|^2 \right)^{\frac{1}{2}}. \quad (1.10)$$

Motivated by the above results, in this paper we introduce the hypo- q -norms on a Cartesian product of inner product spaces. A representation of these norms in terms of inner products, the equivalence with the q -norms on a Cartesian product and some reverse inequalities obtained via the scalar Shisha-Mond, Birnacki et al. and other Grüss type inequalities are also given.

2. GENERAL RESULTS

Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} . We have the following representation result for the *hypo- q -norms* on H^n .

Theorem 3. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} . For any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$, we have*

$$\|\mathbf{x}\|_{h,n,q} = \sup_{\|y\| \leq 1} \left\{ \left(\sum_{j=1}^n |\langle x_j, y \rangle|^q \right)^{1/q} \right\} \quad (2.1)$$

where $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|\mathbf{x}\|_{h,n,1} = \sup_{\|y\| \leq 1} \left\{ \sum_{j=1}^n |\langle x_j, y \rangle| \right\} \quad (2.2)$$

and

$$\|\mathbf{x}\|_{h,n,\infty} = \|\mathbf{x}\|_{n,\infty} = \max_{j \in \{1, \dots, n\}} \{\|x_j\|\}. \quad (2.3)$$

In particular,

$$\|\mathbf{x}\|_{h,e} = \sup_{\|y\| \leq 1} \left\{ \left(\sum_{j=1}^n |\langle x_j, y \rangle|^2 \right)^{1/2} \right\}. \quad (2.4)$$

Proof. Using Hölder's discrete inequality for $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \left(\sum_{j=1}^n |\alpha_j|^p \right)^{1/p} \left(\sum_{j=1}^n |\beta_j|^q \right)^{1/q},$$

which implies that

$$\sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_q \quad (2.5)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with

$$\alpha_j := \frac{\overline{\beta_j} |\beta_j|^{q-2}}{(\sum_{k=1}^n |\beta_k|^q)^{1/p}}$$

for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We observe that

$$\begin{aligned} \left| \sum_{j=1}^n \alpha_j \beta_j \right| &= \left| \sum_{j=1}^n \frac{\overline{\beta_j} |\beta_j|^{q-2}}{(\sum_{k=1}^n |\beta_k|^q)^{1/p}} \beta_j \right| = \frac{\sum_{j=1}^n |\beta_j|^q}{(\sum_{k=1}^n |\beta_k|^q)^{1/p}} \\ &= \left(\sum_{j=1}^n |\beta_j|^q \right)^{1/q} = \|\beta\|_q \end{aligned}$$

and

$$\begin{aligned} \|\alpha\|_p^p &= \sum_{j=1}^n |\alpha_j|^p = \sum_{j=1}^n \frac{|\overline{\beta_j} |\beta_j|^{q-2}|^p}{(\sum_{k=1}^n |\beta_k|^q)^p} = \sum_{j=1}^n \frac{(|\beta_j|^{q-1})^p}{(\sum_{k=1}^n |\beta_k|^q)^p} \\ &= \sum_{j=1}^n \frac{|\beta_j|^{qp-p}}{(\sum_{k=1}^n |\beta_k|^q)^p} = \sum_{j=1}^n \frac{|\beta_j|^q}{(\sum_{k=1}^n |\beta_k|^q)^p} = 1. \end{aligned}$$

Therefore, by (2.5) we have the representation

$$\sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_q \quad (2.6)$$

for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$.

By the properties of inner product, we have for any $u \in H$, $u \neq 0$ that

$$\|u\| = \sup_{\|y\| \leq 1} |\langle u, y \rangle|. \quad (2.7)$$

Let $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{K}^n$ and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then by (2.7) we have

$$\left\| \sum_{j=1}^n \alpha_j x_j \right\| = \sup_{\|y\| \leq 1} \left| \left\langle \sum_{j=1}^n \alpha_j x_j, y \right\rangle \right| = \sup_{\|y\| \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right|. \quad (2.8)$$

By taking the supremum in this equality we have

$$\begin{aligned} \sup_{\|\alpha\|_p \leq 1} \left\| \sum_{j=1}^n \alpha_j x_j \right\| &= \sup_{\|\alpha\|_p \leq 1} \left(\sup_{\|y\| \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right| \right) \\ &= \sup_{\|y\| \leq 1} \left(\sup_{\|\alpha\|_p \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right| \right) = \sup_{\|y\| \leq 1} \left(\sum_{j=1}^n |\langle x_j, y \rangle|^q \right)^{1/2}, \end{aligned}$$

where for the last equality we used the representation (2.6).

This proves (2.1).

Using the properties of the modulus, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \max_{j \in \{1, \dots, n\}} |\alpha_j| \sum_{j=1}^n |\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_\infty \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_1, \quad (2.9)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_j := \frac{\overline{\beta_j}}{|\beta_j|}$ for those j for which $\beta_j \neq 0$ and $\alpha_j = 0$, for the rest.

We have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \sum_{j=1}^n \frac{\overline{\beta_j}}{|\beta_j|} \beta_j \right| = \sum_{j=1}^n |\beta_j| = \|\beta\|_1$$

and

$$\|\alpha\|_\infty = \max_{j \in \{1, \dots, n\}} |\alpha_j| = \max_{j \in \{1, \dots, n\}} \left| \frac{\overline{\beta_j}}{|\beta_j|} \right| = 1$$

and by (2.9) we get the representation

$$\sup_{\|\alpha\|_\infty \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_1 \quad (2.10)$$

for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$.

By taking the supremum in the equality (2.8) we have

$$\begin{aligned} \sup_{\|\alpha\|_\infty \leq 1} \left\| \sum_{j=1}^n \alpha_j x_j \right\| &= \sup_{\|\alpha\|_\infty \leq 1} \left(\sup_{\|y\| \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right| \right) \\ &= \sup_{\|y\| \leq 1} \left(\sup_{\|\alpha\|_\infty \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right| \right) = \sup_{\|y\| \leq 1} \left(\sum_{j=1}^n |\langle x_j, y \rangle| \right), \end{aligned}$$

where for the last equality we used the equality (2.10), which proves the representation (2.2).

Finally, we have

$$\left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \sum_{j=1}^n |\alpha_j| \max_{j \in \{1, \dots, n\}} |\beta_j|,$$

which implies that

$$\sup_{\|\alpha\|_1 \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| \leq \|\beta\|_\infty, \quad (2.11)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$.

For $(\beta_1, \dots, \beta_n) \neq 0$, let $j_0 \in \{1, \dots, n\}$ such that $\|\beta\|_\infty = \max_{j \in \{1, \dots, n\}} |\beta_j| = |\beta_{j_0}|$. Consider $\alpha = (\alpha_1, \dots, \alpha_n)$ with $\alpha_{j_0} = \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|}$ and $\alpha_j = 0$ for $j \neq j_0$.

For this choice we get

$$\sum_{j=1}^n |\alpha_j| = \frac{|\overline{\beta_{j_0}}|}{|\beta_{j_0}|} = 1 \text{ and } \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \left| \frac{\overline{\beta_{j_0}}}{|\beta_{j_0}|} \beta_{j_0} \right| = |\beta_{j_0}| = \|\beta\|_\infty,$$

therefore by (2.11) we obtain the representation

$$\sup_{\|\alpha\|_1 \leq 1} \left| \sum_{j=1}^n \alpha_j \beta_j \right| = \|\beta\|_\infty \quad (2.12)$$

for any $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{K}^n$.

By taking the supremum in the equality (2.8) and by using the equality (2.12), we have

$$\begin{aligned} \sup_{\|\alpha\|_1 \leq 1} \left\| \sum_{j=1}^n \alpha_j x_j \right\| &= \sup_{\|\alpha\|_1 \leq 1} \left(\sup_{\|y\| \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right| \right) \\ &= \sup_{\|y\| \leq 1} \left(\sup_{\|\alpha\|_1 \leq 1} \left| \sum_{j=1}^n \alpha_j \langle x_j, y \rangle \right| \right) = \sup_{\|y\| \leq 1} \left(\max_{j \in \{1, \dots, n\}} |\langle x_j, y \rangle| \right) \\ &= \max_{j \in \{1, \dots, n\}} \left(\sup_{\|y\| \leq 1} |\langle x_j, y \rangle| \right) = \max_{j \in \{1, \dots, n\}} \{\|x_j\|\}, \end{aligned}$$

which proves (2.3). For the last equality we used the property (2.7). \square

Corollary 3.1. *With the assumptions of Theorem 3 we have for $q \geq 1$ that*

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{n,q} \quad (2.13)$$

for any $\mathbf{x} \in H^n$.

In particular, we have

$$\frac{1}{\sqrt{n}} \|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_2 \quad (2.14)$$

for any $\mathbf{x} \in H^n$.

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$ with $\|y\| \leq 1$, then for $q \geq 1$

$$\left(\sum_{j=1}^n |\langle x_j, y \rangle|^q \right)^{1/q} \leq \left(\sum_{j=1}^n (\|y\| \|x_j\|)^q \right)^{1/q} = \|y\| \left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q} = \|y\| \|\mathbf{x}\|_{n,q}$$

and by taking the supremum over $\|y\| \leq 1$, we get the second inequality in (2.13).

By the properties of complex numbers, we have

$$\max_{j \in \{1, \dots, n\}} \{|\langle x_j, y \rangle|\} \leq \left(\sum_{j=1}^n |\langle x_j, y \rangle|^q \right)^{1/q}$$

and by taking the supremum over $\|y\| \leq 1$, we get

$$\sup_{\|y\| \leq 1} \left(\max_{j \in \{1, \dots, n\}} \{|\langle x_j, y \rangle|\} \right) \leq \sup_{\|y\| \leq 1} \left(\sum_{j=1}^n |\langle x_j, y \rangle|^q \right)^{1/q} \quad (2.15)$$

and since

$$\begin{aligned} \sup_{\|y\| \leq 1} \left(\max_{j \in \{1, \dots, n\}} \{|\langle x_j, y \rangle|\} \right) &= \max_{j \in \{1, \dots, n\}} \left\{ \sup_{\|y\| \leq 1} |\langle x_j, y \rangle| \right\} \\ &= \max_{j \in \{1, \dots, n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}, \end{aligned}$$

then by (2.15) we get

$$\|\mathbf{x}\|_{n,\infty} \leq \|\mathbf{x}\|_{h,n,q} \quad \text{for any } \mathbf{x} \in H^n. \quad (2.16)$$

Since

$$\left(\sum_{j=1}^n \|x_j\|^q \right)^{1/q} \leq \left(n \|\mathbf{x}\|_{n,\infty}^q \right)^{1/q} = n^{1/q} \|\mathbf{x}\|_{n,\infty}$$

then also

$$\frac{1}{n^{1/q}} \|\mathbf{x}\|_{n,q} \leq \|\mathbf{x}\|_{n,\infty} \quad \text{for any } \mathbf{x} \in H^n. \quad (2.17)$$

By utilising the inequalities (2.16) and (2.17) we obtain the first inequality in (2.13). \square

Remark 2.1: In the case of inner product spaces the inequality (2.14) has been obtained in a different and more difficult way in [9] by employing the rotation-invariant normalised positive Borel measure on the unit sphere.

Corollary 3.2. *With the assumptions of Theorem 3 we have for $r \geq q \geq 1$ that*

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{r-q}{rq}} \|\mathbf{x}\|_{h,n,r} \quad (2.18)$$

for any $\mathbf{x} \in H^n$.

In particular, for $q \geq 2$ we have

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,e} \leq n^{\frac{q-2}{2q}} \|\mathbf{x}\|_{h,n,q} \quad (2.19)$$

and for $1 \leq q \leq 2$ we have

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,q} \leq n^{\frac{2-q}{2q}} \|\mathbf{x}\|_{h,e} \quad (2.20)$$

for any $\mathbf{x} \in H^n$.

Proof. We use the following elementary inequalities for the nonnegative numbers a_j , $j = 1, \dots, n$ and $r \geq q > 0$ (see for instance [16])

$$\left(\sum_{j=1}^n a_j^r \right)^{1/r} \leq \left(\sum_{j=1}^n a_j^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^n a_j^r \right)^{1/r}. \quad (2.21)$$

Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$ with $\|y\| \leq 1$, then for $r \geq q \geq 1$ we have

$$\left(\sum_{j=1}^n |\langle x_j, y \rangle|^r \right)^{1/r} \leq \left(\sum_{j=1}^n |\langle x_j, y \rangle|^q \right)^{1/q} \leq n^{\frac{r-q}{rq}} \left(\sum_{j=1}^n |\langle x_j, y \rangle|^r \right)^{1/r}. \quad (2.22)$$

By taking the supremum over $y \in H$ with $\|y\| \leq 1$ and using Theorem 3, we get (2.18). \square

Remark 2.2: If we take $q = 1$ in (2.18), then we get for $r \geq 1$ that

$$\|\mathbf{x}\|_{h,n,r} \leq \|\mathbf{x}\|_{h,n,1} \leq n^{\frac{r-1}{r}} \|\mathbf{x}\|_{h,n,r} \quad (2.23)$$

for any $\mathbf{x} \in H^n$.

In particular, for $r = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \|\mathbf{x}\|_{h,n,1} \leq \sqrt{n} \|\mathbf{x}\|_{h,e} \quad (2.24)$$

for any $\mathbf{x} \in H^n$.

3. SOME REVERSE INEQUALITIES

Recall the following additive reverse of Cauchy-Buniakowski-Schwarz inequality [7] (see also [8, Theorem 5. 14])

Lemma 1. Let $a, A \in \mathbb{R}$ and $\mathbf{z} = (z_1, \dots, z_n)$, $\mathbf{y} = (y_1, \dots, y_n)$ be two sequences of real numbers with the property that:

$$ay_j \leq z_j \leq Ay_j \text{ for each } j \in \{1, \dots, n\}. \quad (3.1)$$

Then for any $\mathbf{w} = (w_1, \dots, w_n)$ a sequence of positive real numbers, one has the inequality

$$0 \leq \sum_{j=1}^n w_j z_j^2 \sum_{j=1}^n w_j y_j^2 - \left(\sum_{j=1}^n w_j z_j y_j \right)^2 \leq \frac{1}{4} (A - a)^2 \left(\sum_{j=1}^n w_j y_j^2 \right)^2. \quad (3.2)$$

The constant $\frac{1}{4}$ is sharp in (3.2).

O. Shisha and B. Mond obtained in 1967 (see [17]) the following counterparts of (CBS)- inequality (see also [8, Theorem 5.20 & 5.21])

Lemma 2. Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists a, A, b, B with the property that:

$$0 \leq a \leq a_j \leq A \text{ and } 0 < b \leq b_j \leq B \text{ for any } j \in \{1, \dots, n\}, \quad (3.3)$$

then we have the inequality

$$\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 - \left(\sum_{j=1}^n a_j b_j \right)^2 \leq \left(\sqrt{\frac{A}{b}} - \sqrt{\frac{a}{B}} \right)^2 \sum_{j=1}^n a_j b_j \sum_{j=1}^n b_j^2, \quad (3.4)$$

and

Lemma 3. Assume that \mathbf{a}, \mathbf{b} are nonnegative sequences and there exists γ, Γ with the property that

$$0 \leq \gamma \leq \frac{a_j}{b_j} \leq \Gamma < \infty \text{ for any } j \in \{1, \dots, n\}. \quad (3.5)$$

Then we have the inequality

$$0 \leq \left(\sum_{j=1}^n a_j^2 \sum_{j=1}^n b_j^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n a_j b_j \leq \frac{(\Gamma - \gamma)^2}{4(\gamma + \Gamma)} \sum_{j=1}^n b_j^2. \quad (3.6)$$

We have the following result:

Theorem 4. Let $(H, \|\cdot\|)$ be an inner product space over the real or complex number field \mathbb{K} and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then we have

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^2, \quad (3.7)$$

$$0 \leq \|\mathbf{x}\|_{h,e}^2 - \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 \leq \|\mathbf{x}\|_{h,n,1} \|\mathbf{x}\|_{n,\infty} \quad (3.8)$$

and

$$0 \leq \|\mathbf{x}\|_{h,e} - \frac{1}{\sqrt{n}} \|\mathbf{x}\|_{h,n,1} \leq \frac{1}{4} \sqrt{n} \|\mathbf{x}\|_{n,\infty}. \quad (3.9)$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and put $R = \max_{j \in \{1, \dots, n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n,\infty}$. If $y \in H$ with $\|y\| \leq 1$ then $|\langle x_j, y \rangle| \leq \|y\| \|x_j\| \leq R$ for any $j \in \{1, \dots, n\}$.

If we write the inequality (3.2) for $z_j = |\langle x_j, y \rangle|$, $w_j = y_j = 1$, $A = R$ and $a = 0$, we get

$$0 \leq n \sum_{j=1}^n |\langle x_j, y \rangle|^2 - \left(\sum_{j=1}^n |\langle x_j, y \rangle| \right)^2 \leq \frac{1}{4} n^2 R^2$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\sum_{j=1}^n |\langle x_j, y \rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |\langle x_j, y \rangle| \right)^2 + \frac{1}{4} n R^2 \quad (3.10)$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.10) over $y \in H$ with $\|y\| \leq 1$ we get (3.7).

If we write the inequality (3.4) for $a_j = |\langle x_j, y \rangle|$, $b_j = 1$, $b = B = 1$, $a = 0$ and $A = R$, then we get

$$0 \leq n \sum_{j=1}^n |\langle x_j, y \rangle|^2 - \left(\sum_{j=1}^n |\langle x_j, y \rangle| \right)^2 \leq nR \sum_{j=1}^n |\langle x_j, y \rangle|,$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\sum_{j=1}^n |\langle x_j, y \rangle|^2 \leq \frac{1}{n} \left(\sum_{j=1}^n |\langle x_j, y \rangle| \right)^2 + R \sum_{j=1}^n |\langle x_j, y \rangle|, \quad (3.11)$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.11) over $y \in H$ with $\|y\| \leq 1$ we get (3.8).

Finally, if we write the inequality (3.6) for $a_j = |\langle x_j, y \rangle|$, $b_j = 1$, $b = B = 1$, $\gamma = 0$ and $\Gamma = R$, then we have

$$0 \leq \left(n \sum_{j=1}^n |\langle x_j, y \rangle|^2 \right)^{\frac{1}{2}} - \sum_{j=1}^n |\langle x_j, y \rangle| \leq \frac{1}{4} nR,$$

for any $y \in H$ with $\|y\| \leq 1$.

This implies that

$$\left(\sum_{j=1}^n |\langle x_j, y \rangle|^2 \right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{n}} \sum_{j=1}^n |\langle x_j, y \rangle| + \frac{1}{4} \sqrt{n} R, \quad (3.12)$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.12) over $y \in H$ with $\|y\| \leq 1$ we get (3.9). \square

Further, we recall the *Čebyšev's inequality* for *synchronous* n -tuples of vectors $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$, namely if $(a_j - a_k)(b_j - b_k) \geq 0$ for any $j, k \in \{1, \dots, n\}$, then

$$\frac{1}{n} \sum_{j=1}^n a_j b_j \geq \frac{1}{n} \sum_{j=1}^n a_j \frac{1}{n} \sum_{j=1}^n b_j. \quad (3.13)$$

In 1950, Biernacki et al. [2] obtained the following discrete version of Grüss' inequality:

Lemma 4. *Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are such that there exists real numbers a, A, b, B with the property that:*

$$a \leq a_j \leq A \quad \text{and} \quad b \leq b_j \leq B \quad \text{for any } j \in \{1, \dots, n\}. \quad (3.14)$$

Then

$$\begin{aligned} & \left| \frac{1}{n} \sum_{j=1}^n a_j b_j - \frac{1}{n} \sum_{j=1}^n a_j \frac{1}{n} \sum_{j=1}^n b_j \right| \\ & \leq \frac{1}{n} \left[\frac{n}{2} \right] \left(1 - \frac{1}{n} \left[\frac{n}{2} \right] \right) (A - a)(B - b) \\ & = \frac{1}{n^2} \left[\frac{n^2}{4} \right] (A - a)(B - a) \leq \frac{1}{4} (A - a)(B - b), \end{aligned} \quad (3.15)$$

where $[x]$ gives the largest integer less than or equal to x .

The following result also holds:

Theorem 5. *Let $(H, \|\cdot\|)$ be an inner product space over the real or complex number field \mathbb{K} and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q, r \geq 1$ we have*

$$\begin{aligned} \|\mathbf{x}\|_{h,n,q+r}^{q+r} & \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^q \|\mathbf{x}\|_{h,n,r}^r + \frac{1}{n} \left[\frac{n^2}{4} \right] \|\mathbf{x}\|_{n,\infty}^{q+r} \\ & \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,q}^q \|\mathbf{x}\|_{h,n,r}^r + \frac{1}{4} n \|\mathbf{x}\|_{n,\infty}^{q+r}. \end{aligned} \quad (3.16)$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and put $R = \max_{j \in \{1, \dots, n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n, \infty}$. If $y \in H$ with $\|y\| \leq 1$ then $|\langle x_j, y \rangle| \leq \|y\| \|x_j\| \leq R$ for any $j \in \{1, \dots, n\}$.

If we take into the inequality (3.15) $a_j = |\langle x_j, y \rangle|^q$, $b_j = |\langle x_j, y \rangle|^r$, $a = 0$, $A = R^q$, $b = 0$ and $B = R^r$, then we get

$$\left| \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} - \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^r \right| \leq \frac{1}{n^2} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r}. \quad (3.17)$$

On the other hand, since the sequences $\{a_j\}_{j=1, \dots, n}$, $\{b_j\}_{j=1, \dots, n}$ are synchronous, then by (3.13) we have

$$0 \leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} - \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^r.$$

Using (3.17) we then get

$$\sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} \leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r} \quad (3.18)$$

for any $y \in H$ with $\|y\| \leq 1$.

By taking the supremum in (3.18), we get

$$\begin{aligned} & \sup_{\|y\| \leq 1} \left\{ \sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} \right\} \\ & \leq \frac{1}{n} \sup_{\|y\| \leq 1} \left\{ \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r \right\} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r} \\ & \leq \frac{1}{n} \sup_{\|y\| \leq 1} \left\{ \sum_{j=1}^n |\langle x_j, y \rangle|^q \right\} \sup_{\|y\| \leq 1} \left\{ \sum_{j=1}^n |\langle x_j, y \rangle|^r \right\} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil R^{q+r}, \end{aligned}$$

which proves the first inequality in (3.16).

The second part of (3.16) is obvious. \square

Corollary 5.1. *With the assumptions of Theorem 5 and if $r \geq 1$, then we have*

$$\|\mathbf{x}\|_{h, n, 2r}^{2r} \leq \frac{1}{n} \|\mathbf{x}\|_{h, n, r}^{2r} + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n, \infty}^{2r} \leq \frac{1}{n} \|\mathbf{x}\|_{h, n, r}^{2r} + \frac{1}{4} n \|\mathbf{x}\|_{n, \infty}^{2r}. \quad (3.19)$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h, e}^2 \leq \frac{1}{n} \|\mathbf{x}\|_{h, n, 1}^2 + \frac{1}{n} \left\lceil \frac{n^2}{4} \right\rceil \|\mathbf{x}\|_{n, \infty}^2 \leq \frac{1}{n} \|\mathbf{x}\|_{h, n, 1}^2 + \frac{1}{4} n \|\mathbf{x}\|_{n, \infty}^2. \quad (3.20)$$

The first inequality in (3.20) is better than the second inequality in (3.7).

For an n -tuple of complex numbers $\mathbf{a} = (a_1, \dots, a_n)$ with $n \geq 2$ consider the $(n-1)$ -tuple built by the aid of forward differences $\Delta \mathbf{a} = (\Delta a_1, \dots, \Delta a_{n-1})$ where $\Delta a_k := a_{k+1} - a_k$ where $k \in \{1, \dots, n-1\}$. Similarly, if $\mathbf{x} = (x_1, \dots, x_n) \in H^n$ is an n -tuple of vectors we also can consider in a similar way the $(n-1)$ -tuple $\Delta \mathbf{x} = (\Delta x_1, \dots, \Delta x_{n-1})$.

We obtained the following Grüss' type inequalities in terms of forward differences:

Lemma 5. *Assume that $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$ are n -tuples of complex numbers. Then*

$$\begin{aligned} & \left| \frac{1}{n} \sum_{j=1}^n a_j b_j - \frac{1}{n} \sum_{j=1}^n a_j \frac{1}{n} \sum_{j=1}^n b_j \right| \tag{3.21} \\ & \leq \begin{cases} \frac{1}{12} (n^2 - 1) \|\Delta \mathbf{a}\|_{n-1, \infty} \|\Delta \mathbf{b}\|_{n-1, \infty}, & [12], \\ \frac{1}{6} \frac{n^2 - 1}{n} \|\Delta \mathbf{a}\|_{n-1, \alpha} \|\Delta \mathbf{b}\|_{n-1, \beta} \text{ where } \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, & [5], \\ \frac{1}{2} \left(1 - \frac{1}{n}\right) \|\Delta \mathbf{a}\|_{n-1, 1} \|\Delta \mathbf{b}\|_{n-1, 1}, & [6]. \end{cases} \end{aligned}$$

The constants $\frac{1}{12}$, $\frac{1}{6}$ and $\frac{1}{2}$ are best possible in (3.21).

The following result also holds:

Theorem 6. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q, r \geq 1$ we have*

$$\begin{aligned} \|\mathbf{x}\|_{h, n, q+r}^{q+r} & \leq \frac{1}{n} \|\mathbf{x}\|_{h, n, q}^q \|\mathbf{x}\|_{h, n, r}^r \tag{3.22} \\ & + \begin{cases} \frac{1}{12} q r (n^2 - 1) n \|\mathbf{x}\|_{n, \infty}^{q+r-2} \|\Delta \mathbf{x}\|_{n-1, \infty}^2, \\ \frac{1}{6} (n^2 - 1) q r \|\mathbf{x}\|_{n, \infty}^{q+r-2} \|\Delta \mathbf{x}\|_{h, n-1, \alpha} \|\Delta \mathbf{x}\|_{h, n-1, \beta} \\ \text{where } \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} (n-1) q r \|\mathbf{x}\|_{n, \infty}^{q+r-2} \|\Delta \mathbf{x}\|_{h, n-1, 1}^2. \end{cases} \end{aligned}$$

Proof. Let $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$ with $\|y\| \leq 1$. If we take into the inequality (3.21) $a_j = |\langle x_j, y \rangle|^q$, $b_j = |\langle x_j, y \rangle|^r$, then we get

$$\left| \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} - \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^r \right| \tag{3.23}$$

$$\leq \begin{cases} \frac{1}{12} (n^2 - 1) \max_{j=1, \dots, n-1} |\Delta |\langle x_j, y \rangle|^q| \max_{j=1, \dots, n-1} |\Delta |\langle x_j, y \rangle|^r|, \\ \frac{1}{6} \frac{n^2-1}{n} \left(\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^q|^\alpha \right)^{1/\alpha} \left(\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^r|^\beta \right)^{1/\beta} \\ \text{where } \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} \left(1 - \frac{1}{n}\right) \sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^q| \sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^r|. \end{cases}$$

We use the following elementary inequality for powers $p \geq 1$

$$|a^p - b^p| \leq pR^{p-1} |a - b|$$

where $a, b \in [0, R]$.

Put $R = \max_{j \in \{1, \dots, n\}} \{\|x_j\|\} = \|\mathbf{x}\|_{n, \infty}$. Then for any $y \in H$ with $\|y\| \leq 1$ we have $|\langle x_j, y \rangle| \leq \|y\| \|x_j\| \leq R$ for any $j \in \{1, \dots, n\}$.

Therefore

$$\begin{aligned} |\Delta |\langle x_j, y \rangle|^q| &= |\langle x_{j+1}, y \rangle|^q - |\langle x_j, y \rangle|^q \leq qR^{q-1} |\langle x_{j+1}, y \rangle| - |\langle x_j, y \rangle| \\ &\leq qR^{q-1} |\langle x_{j+1}, y \rangle - \langle x_j, y \rangle| = qR^{q-1} |\langle \Delta x_j, y \rangle| \end{aligned} \quad (3.24)$$

for any $j = 1, \dots, n-1$, where $\Delta x_j = x_{j+1} - x_j$ is the forward difference.

On the other hand, since the sequences $\{a_j\}_{j=1, \dots, n}$, $\{b_j\}_{j=1, \dots, n}$ are synchronous, then we have

$$0 \leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} - \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^r \quad (3.25)$$

and by the first inequality in (3.23) we get

$$\begin{aligned} &\sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} \\ &\leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r \\ &+ \frac{1}{12} (n^2 - 1) nqR^{q-1} \max_{j=1, \dots, n-1} |\langle \Delta x_j, y \rangle| rR^{r-1} \max_{j=1, \dots, n-1} |\langle \Delta x_j, y \rangle| \\ &= \frac{1}{n} \sum_{j=1}^n |\langle \Delta x_j, y \rangle|^q \sum_{j=1}^n |\langle \Delta x_j, y \rangle|^r \\ &+ \frac{1}{12} (n^2 - 1) nqrR^{q+r-2} \left(\max_{j=1, \dots, n-1} |\langle \Delta x_j, y \rangle| \right)^2 \end{aligned} \quad (3.26)$$

for any $y \in H$ with $\|y\| \leq 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.26) we get the first branch in the inequality (3.22).

We also have, by (3.24), that

$$\begin{aligned} \left(\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^q|^\alpha \right)^{1/\alpha} &\leq \left[(qR^{q-1})^\alpha \sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|^\alpha \right]^{1/\alpha} \\ &= qR^{q-1} \left(\sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|^\alpha \right)^{1/\alpha} \end{aligned}$$

and, similarly,

$$\left(\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^r|^\beta \right)^{1/\beta} \leq rR^{r-1} \left(\sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|^\beta \right)^{1/\beta}$$

where $\alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1$.

By the second inequality in (3.23) and by (3.25) we have

$$\begin{aligned} &\sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} \tag{3.27} \\ &\leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r \\ &+ \frac{1}{6} (n^2 - 1) \left(\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^q|^\alpha \right)^{1/\alpha} \left(\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^r|^\beta \right)^{1/\beta} \\ &\leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r \\ &+ \frac{1}{6} (n^2 - 1) qrR^{q+r-2} \left(\sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|^\alpha \right)^{1/\alpha} \left(\sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|^\beta \right)^{1/\beta} \end{aligned}$$

for any $y \in H$ with $\|y\| \leq 1$, where $\alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.27) we get the second branch in the inequality (3.22).

We also have, by (3.24), that

$$\sum_{j=1}^{n-1} |\Delta |\langle x_j, y \rangle|^q| \leq qR^{q-1} \sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|$$

and

$$\sum_{j=1}^{n-1} |\Delta \langle x_j, y \rangle|^r \leq rR^{r-1} \sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle|.$$

By the third inequality in (3.23) and by (3.25) we have

$$\begin{aligned} \sum_{j=1}^n |\langle x_j, y \rangle|^{q+r} &\leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r \\ &+ \frac{1}{2} (n-1) \sum_{j=1}^{n-1} |\Delta \langle x_j, y \rangle|^q \sum_{j=1}^{n-1} |\Delta \langle x_j, y \rangle|^r \\ &\leq \frac{1}{n} \sum_{j=1}^n |\langle x_j, y \rangle|^q \sum_{j=1}^n |\langle x_j, y \rangle|^r \\ &+ \frac{1}{2} (n-1) qrR^{q+r-2} \sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle| \sum_{j=1}^{n-1} |\langle \Delta x_j, y \rangle| \end{aligned} \quad (3.28)$$

for any $y \in H$ with $\|y\| \leq 1$.

Taking the supremum over $y \in H$ with $\|y\| \leq 1$ in (3.28) we get the third branch in the inequality (3.22). \square

Corollary 6.1. *With the assumptions of Theorem 6 and if $r \geq 1$, then we have*

$$\begin{aligned} \|\mathbf{x}\|_{h,n,2r}^{2r} &\leq \frac{1}{n} \|\mathbf{x}\|_{h,n,r}^{2r} \\ &+ \begin{cases} \frac{1}{12} r^2 (n^2 - 1) n \|\mathbf{x}\|_{n,\infty}^{2r-2} \|\Delta \mathbf{x}\|_{n-1,\infty}^2, \\ \frac{1}{6} r^2 (n^2 - 1) \|\mathbf{x}\|_{n,\infty}^{2r-2} \|\Delta \mathbf{x}\|_{h,n-1,\alpha} \|\Delta \mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} r^2 (n-1) \|\mathbf{x}\|_{n,\infty}^{2r-2} \|\Delta \mathbf{x}\|_{h,n-1,1}^2. \end{cases} \end{aligned} \quad (3.29)$$

In particular, for $r = 1$ we get

$$\|\mathbf{x}\|_{h,e}^2 \leq \frac{1}{n} \|\mathbf{x}\|_{h,n,1}^2 + \begin{cases} \frac{1}{12} (n^2 - 1) n \|\Delta \mathbf{x}\|_{n-1,\infty}^2, \\ \frac{1}{6} (n^2 - 1) \|\Delta \mathbf{x}\|_{h,n-1,\alpha} \|\Delta \mathbf{x}\|_{h,n-1,\beta} \\ \text{where } \alpha, \beta > 1, \frac{1}{\alpha} + \frac{1}{\beta} = 1, \\ \frac{1}{2} (n-1) \|\Delta \mathbf{x}\|_{h,n-1,1}^2. \end{cases} \quad (3.30)$$

4. FURTHER INEQUALITIES

In 1992, J. Pečarić [15] proved the following general inequality in inner product spaces:

Lemma 6. *Let $y, x_1, \dots, x_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then*

$$\begin{aligned} \left| \sum_{i=1}^n c_i \langle x_i, y \rangle \right|^2 &\leq \|y\|^2 \sum_{i=1}^n |c_i|^2 \left(\sum_{j=1}^n |\langle x_i, x_j \rangle| \right) \\ &\leq \|y\|^2 \sum_{i=1}^n |c_i|^2 \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\}. \end{aligned} \quad (4.1)$$

He showed that the Bombieri inequality [4] may be obtained from (4.1) for the choice $c_i = \overline{\langle x_i, y \rangle}$ (using the second inequality), the Selberg inequality [14, p. 394] may be obtained from the first part of (4.1) for the choice

$$c_i = \frac{\overline{\langle x_i, y \rangle}}{\sum_{j=1}^n |\langle x_i, x_j \rangle|}, \quad i \in \{1, \dots, n\};$$

while the Heilbronn inequality [13] may be obtained from the first part of (4.1) if one chooses $c_i = \frac{\langle x_i, y \rangle}{|\langle x_i, y \rangle|}$, for any $i \in \{1, \dots, n\}$.

Theorem 7. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q \geq 1$ we have*

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q} \left[\max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\} \right]^{\frac{1}{2q}}. \quad (4.2)$$

In particular, for $q = 1$ we get

$$\|\mathbf{x}\|_{h,n,1} \leq \left[\max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\} \right]^{\frac{1}{2}} \quad (4.3)$$

while for $q = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \left[\max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\} \right]^{\frac{1}{2}}. \quad (4.4)$$

Proof. If we take in (4.1) $c_i = \overline{\langle x_i, y \rangle} |\langle x_i, y \rangle|^{q-2}$, $i \in \{1, \dots, n\}$, then we get

$$\left(\sum_{i=1}^n |\langle x_i, y \rangle|^q \right)^2 \leq \|y\|^2 \sum_{i=1}^n |\langle x_i, y \rangle|^{2(q-1)} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\} \quad (4.5)$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

By taking the square root in (4.5) we get

$$\begin{aligned} & \sum_{i=1}^n |\langle x_i, y \rangle|^q \\ & \leq \|y\| \left(\sum_{i=1}^n |\langle x_i, y \rangle|^{2(q-1)} \right)^{1/2} \left[\max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\} \right]^{1/2}, \end{aligned} \quad (4.6)$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

If we take the supremum in (4.6) for $\|y\| \leq 1$, then we get

$$\|\mathbf{x}\|_{h,n,q}^q \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{q-1} \left[\max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n |\langle x_i, x_j \rangle| \right\} \right]^{1/2}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

This proves (4.2). \square

In 1941, R. P. Boas [3] and in 1944, independently, R. Bellman [1] proved the following generalization of Bessel's inequality

Lemma 7. *If y, x_1, \dots, x_n are elements of an inner product space $(H; \langle \cdot, \cdot \rangle)$, then the following inequality:*

$$\sum_{i=1}^n |\langle x_i, y \rangle|^2 \leq \|y\|^2 \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right], \quad (4.7)$$

holds.

A generalization of the Boas-Bellman result was given in Mitrinović-Pečarić-Fink [14, p. 392] where they proved the following:

Lemma 8. *If y, x_1, \dots, x_n are as in Lemma 7 and $c_1, \dots, c_n \in \mathbb{K}$, then one has the inequality:*

$$\left| \sum_{i=1}^n c_i \langle x_i, y \rangle \right|^2 \leq \|y\|^2 \sum_{i=1}^n |c_i|^2 \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]. \quad (4.8)$$

They also noted that if in (4.8) one chooses $c_i = \overline{\langle x, y_i \rangle}$, then this inequality becomes (4.7).

Using a similar argument to the one in Theorem 7 and Lemma 8 we have:

Theorem 8. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q \geq 1$ we have

$$\|\mathbf{x}\|_{h,n,q} \leq \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q} \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2q}}. \quad (4.9)$$

In particular, for $q = 1$ we get

$$\|\mathbf{x}\|_{h,n,1} \leq \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}} \quad (4.10)$$

while for $q = 2$ we get

$$\|\mathbf{x}\|_{h,e} \leq \left[\max_{1 \leq i \leq n} \|x_i\|^2 + \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^2 \right)^{\frac{1}{2}} \right]^{\frac{1}{2}}. \quad (4.11)$$

In [10] we obtained the following result that provides some companions to the Boas-Bellman inequality above:

Lemma 9. Let $y, x_1, \dots, x_n \in H$ and $c_1, \dots, c_n \in \mathbb{K}$. Then

$$\left| \sum_{i=1}^n c_i \langle x_i, y \rangle \right|^2 \quad (4.12)$$

$$\leq \|y\|^2 \times \begin{cases} \max_{1 \leq i \leq n} \{ |c_i|^2 \} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}; \\ \left(\sum_{i=1}^n |c_i|^{2p} \right)^{\frac{1}{p}} \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \sum_{i=1}^n |c_i|^2 \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}. \end{cases}$$

We have:

Theorem 9. Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space over the real or complex number field \mathbb{K} and $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$. Then for $q \geq 1$ we

have

$$\|\mathbf{x}\|_{h,n,q} \leq \begin{cases} \max_{1 \leq i \leq n} \{\|x_i\|\}^{1-1/q} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{\frac{1}{2q}}; \\ \|\mathbf{x}\|_{h,n,2p(q-1)}^{1-1/q} \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}^{\frac{1}{2q}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \|\mathbf{x}\|_{h,n,2(q-1)}^{1-1/q} \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{\frac{1}{2q}}. \end{cases} \quad (4.13)$$

In particular, we have for $q = 1$ that

$$\|\mathbf{x}\|_{h,n,1} \leq \begin{cases} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{\frac{1}{2}}; \\ \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}^{\frac{1}{2}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{\frac{1}{2}}, \end{cases} \quad (4.14)$$

while for $q = 2$ that

$$\|\mathbf{x}\|_{h,e}^2 \leq \begin{cases} \max_{1 \leq i \leq n} \{\|x_i\|\} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{\frac{1}{2}}; \\ \|\mathbf{x}\|_{h,n,2p} \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}^{\frac{1}{2}}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \|\mathbf{x}\|_{h,e} \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{\frac{1}{2}}. \end{cases} \quad (4.15)$$

Proof. If we take in (4.12) $c_i = \overline{\langle x_i, y \rangle} |\langle x_i, y \rangle|^{q-2}$, $i \in \{1, \dots, n\}$, then we get

$$\begin{aligned} & \left(\sum_{i=1}^n |\langle x_i, y \rangle|^q \right)^2 \\ & \leq \|y\|^2 \times \left\{ \begin{array}{l} \max_{1 \leq i \leq n} \left\{ |\langle x_i, y \rangle|^{2(q-1)} \right\} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}; \\ \left(\sum_{i=1}^n |\langle x_i, y \rangle|^{2p(q-1)} \right)^{\frac{1}{p}} \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \sum_{i=1}^n |\langle x_i, y \rangle|^{2(q-1)} \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}, \end{array} \right. \end{aligned} \quad (4.16)$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

If we take the square root in this inequality, then we get

$$\begin{aligned} & \sum_{i=1}^n |\langle x_i, y \rangle|^q \\ & \leq \|y\| \times \left\{ \begin{array}{l} \max_{1 \leq i \leq n} \left\{ |\langle x_i, y \rangle|^{q-1} \right\} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{1/2}; \\ \left(\sum_{i=1}^n |\langle x_i, y \rangle|^{2p(q-1)} \right)^{\frac{1}{2p}} \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}^{1/2}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \left(\sum_{i=1}^n |\langle x_i, y \rangle|^{2(q-1)} \right)^{1/2} \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{1/2}, \end{array} \right. \end{aligned} \quad (4.17)$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$ and $y \in H$.

If we take the supremum over $y \in H$, $\|y\| \leq 1$, then we get

$$\|\mathbf{x}\|_{h,n,q}^q \leq \begin{cases} \max_{1 \leq i \leq n} \left\{ \|x_i\|^{q-1} \right\} \left\{ \sum_{i=1}^n \|x_i\|^2 + \sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{1/2}; \\ \|\mathbf{x}\|_{h,n,2p(q-1)}^{q-1} \left\{ \left(\sum_{i=1}^n \|x_i\|^{2r} \right)^{\frac{1}{r}} \right. \\ \left. + (n-1)^{\frac{1}{p}} \left(\sum_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle|^r \right)^{\frac{1}{r}} \right\}^{1/2}, \\ \text{where } p > 1, \frac{1}{p} + \frac{1}{r} = 1; \\ \|\mathbf{x}\|_{h,n,2(q-1)}^{q-1} \left\{ \max_{1 \leq i \leq n} \|x_i\|^2 + (n-1) \max_{1 \leq i \neq j \leq n} |\langle x_i, x_j \rangle| \right\}^{1/2}, \end{cases}$$

for any $\mathbf{x} \in H^n$ with $\mathbf{x} = (x_1, \dots, x_n)$, which produces the desired result (4.13). \square

The interested reader may obtain other similar results by utilising the following inequalities obtained in [11]

Lemma 10. *Let y, x_1, \dots, x_n be vectors of an inner product space $(H; \langle \cdot, \cdot \rangle)$ and $c_1, \dots, c_n \in \mathbb{K}$. Then one has the inequalities:*

$$\left| \sum_{i=1}^n c_i \langle x_i, y \rangle \right|^2 \leq \|y\|^2 \times \begin{cases} D \\ E \\ F \end{cases}, \quad (4.18)$$

where

$$D := \begin{cases} \max_{1 \leq k \leq n} |c_k|^2 \sum_{i,j=1}^n |\langle x_i, x_j \rangle|; \\ \max_{1 \leq k \leq n} |c_k| \left(\sum_{i=1}^n |c_i|^r \right)^{\frac{1}{r}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n |\langle x_i, x_j \rangle| \right)^s \right]^{\frac{1}{s}}, \quad r > 1, \frac{1}{r} + \frac{1}{s} = 1; \\ \max_{1 \leq k \leq n} |c_k| \sum_{k=1}^n |c_k| \max_{1 \leq i \leq n} \left(\sum_{j=1}^n |\langle x_i, x_j \rangle| \right); \end{cases}$$

$$E := \begin{cases} \left(\sum_{k=1}^n |c_k|^p \right)^{\frac{1}{p}} \max_{1 \leq i \leq n} |c_i| \left(\sum_{i=1}^n \left(\sum_{j=1}^n |\langle x_i, x_j \rangle| \right)^q \right)^{\frac{1}{q}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ \left(\sum_{k=1}^n |c_k|^p \right)^{\frac{1}{p}} \left(\sum_{i=1}^n |c_i|^t \right)^{\frac{1}{t}} \left[\sum_{i=1}^n \left(\sum_{j=1}^n |\langle x_i, x_j \rangle|^q \right)^{\frac{u}{q}} \right]^{\frac{1}{u}}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \\ & t > 1, \frac{1}{t} + \frac{1}{u} = 1; \\ \left(\sum_{k=1}^n |c_k|^p \right)^{\frac{1}{p}} \sum_{i=1}^n |c_i| \max_{1 \leq i \leq n} \left\{ \left(\sum_{j=1}^n |\langle x_i, x_j \rangle|^q \right)^{\frac{1}{q}} \right\}, & p > 1, \frac{1}{p} + \frac{1}{q} = 1; \end{cases}$$

and

$$F := \begin{cases} \sum_{k=1}^n |c_k| \max_{1 \leq i \leq n} |c_i| \sum_{i=1}^n \left[\max_{1 \leq j \leq n} |\langle x_i, x_j \rangle| \right]; \\ \sum_{k=1}^n |c_k| \left(\sum_{i=1}^n |c_i|^m \right)^{\frac{1}{m}} \left(\sum_{i=1}^n \left[\max_{1 \leq j \leq n} |\langle x_i, x_j \rangle| \right]^l \right)^{\frac{1}{l}}, & m > 1, \frac{1}{m} + \frac{1}{l} = 1; \\ \left(\sum_{k=1}^n |c_k| \right)^2 \max_{i, 1 \leq j \leq n} |\langle x_i, x_j \rangle|. \end{cases}$$

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